Analyse et Conception Formelles

Lesson 2

–

Types, terms and functions

Outline

1. Terms
   - Types
   - Typed terms
   - λ-terms
   - Constructor terms
2. Functions defined using equations
   - Logic everywhere!
   - Function evaluation using term rewriting
   - Partial functions

Acknowledgements: some slides are borrowed from T. Nipkow's lectures

Types: syntax

τ ::=	(τ)	
|	*bool* | *nat* | *char* | ...	base types
|	*'a* | *'b* | ...	type variables
|	$\tau \Rightarrow \tau$	functions
|	$\tau \times \ldots \times \tau$	tuples (ascii for $\times$: $*$)
|	τ *list*	lists
|	...	user-defined types

The operator $\Rightarrow$ is right-associative, for instance:

$$nat \Rightarrow nat \Rightarrow bool \text{ is equivalent to } nat \Rightarrow (nat \Rightarrow bool)$$

Typed terms: syntax

term ::=	$(term)$	
|	a	$a \in \mathcal{F}$ or $a \in \mathcal{X}$
|	*term term*	function application
|	$\lambda y.\ term$	function definition with $y \in \mathcal{X}$
|	$(term, \ldots, term)$	tuples
|	$[term, \ldots, term]$	lists
|	$(term :: \tau)$	type annotation
|	...	a lot of syntactic sugar

Function application is left-associative, for instance:

$$f\ a\ b\ c \text{ is equivalent to } ((f\ a)\ b)\ c$$

Example 1 (Types of terms)

Term	Type	Term	Type
y	'a	t1	'a
(t1,t2,t3)	('a × 'b × 'c)	[t1,t2,t3]	'a list
λ y. y	'a ⇒ 'a	λ y z. z	'a ⇒ 'b ⇒ 'b

Types and terms: evaluation in Isabelle/HOL

To evaluate a term t in Isabelle .... value "t"

Example 2

Term	Isabelle's answer
value "True"	True::bool
value "2"	Error (cannot infer result type)
value "(2::nat)"	2::nat
value "[True,False]"	[True,False]::bool list
value "(True,True,False)"	(True,True,False)::bool * bool * bool
value "[2,6,10]"	Error (cannot infer result type)
value "[(2::nat),6,10]"	[2,6,10]::nat list

Terms and functions: semantics is the λ-calculus

Semantics of functional programming languages consists of one rule:

$$(\lambda\, x.\, t)\ a \ \twoheadrightarrow_\beta\ t\{x \mapsto a\} \qquad (\beta\text{-reduction})$$

where $t\{x \mapsto a\}$ is the term t where all occurrences of x are replaced by a

Example 3

- $(\lambda\, x.\, x + 1)\ 10 \ \twoheadrightarrow_\beta\ 10 + 1$
- $(\lambda\, x.\lambda\, y.\, x + y)\ 1\ 2 \ \twoheadrightarrow_\beta\ (\lambda\, y.\, 1 + y)\ 2 \ \twoheadrightarrow_\beta\ 1 + 2$
- $(\lambda\, (x, y).\, y)\ (1, 2) \ \twoheadrightarrow_\beta\ 2$

In Isabelle/HOL, to be β-reduced, terms have to be well-typed

Example 4

Previous examples can be reduced because:

- $(\lambda\, x.\, x + 1) :: nat \Rightarrow nat$ and $10 :: nat$
- $(\lambda\, x.\lambda\, y.\, x + y) :: nat \Rightarrow nat \Rightarrow nat$ and $1 :: nat$ and $2 :: nat$
- $(\lambda\, (x, y).y) :: (\text{'a} \times \text{'b}) \Rightarrow \text{'b}$ and $(1, 2) :: nat \times nat$

Lambda-calculus – the quiz

Quiz 1

- *Function $\lambda(x, y).\, x$ is a function with two parameters*

V	True	R	False

- *Type of function $\lambda(x, y).\, x$ is*

V	`'a × 'b ⇒ 'a`
R	`'a ⇒ 'b ⇒ 'a`

- *If* `f::nat ⇒ nat ⇒ nat` *how to call* `f` *on* 1 *and* 2*?*

V	`f(1,2)`	R	`(f 1 2)`

- *If* `f::nat × nat ⇒ nat` *how to call* `f` *on* 1 *and* 2*?*

V	`f(1,2)`	R	`(f 1 2)`

A word about curried functions and partial application

Definition 5 (Curried function)

A function is *curried* if it returns a function as result.

Example 6

The function $(\lambda\, x.\lambda\, y.\, x + y) :: nat \Rightarrow nat \Rightarrow nat$ is curried
The function $(\lambda\, (x, y).\, x + y) :: nat \times nat \Rightarrow nat$ is *not* curried

Example 7 (Curried function can be partially applied!)

The function $(\lambda\, x.\lambda\, y.\, x + y)$ can be applied to 2 or 1 argument!

- $(\lambda\, x.\lambda\, y.\, x + y)\ 1\ 2 \ \twoheadrightarrow_\beta\ (\lambda\, y.\, 1 + y)\ 2 \ \twoheadrightarrow_\beta\ (1 + 2) :: nat$
- $(\lambda\, x.\lambda\, y.\, x + y)\ 1 \ \twoheadrightarrow_\beta (\lambda\, y.\, 1 + y) :: nat \Rightarrow nat$ which is a function!

Exercise 1 (In Isabelle/HOL)

Use `append::'a list ⇒ 'a list ⇒ 'a list` *to concatenate 2 lists of bool, 2 lists of nat, and 3 lists of nat.*

A word about curried functions and partial application (II)

- To associate the value of a term t to a name n . . . . . . definition "n=t"

Exercise 2 (In Isabelle/HOL)

1. *Define the (non-curried) function* addNc *adding two naturals*
2. *Use* addNc *to add* 5 *to* 6
3. *Define the (curried) function* add *adding two naturals*
4. *Use* add *to add* 5 *to* 6
5. *Using* add*, define the* incr *function adding* 1 *to a natural*
6. *Apply* incr *to* 5
7. *Define a function* app1 *adding* 1 *at the beginning of any list of naturals, give an example of use*

A word about higher-order functions

Definition 8 (Higher-order function)

A *higher-order* function takes one or more functions as parameters.

Example 9 (Some higher-order functions and their evaluation)

- $\lambda x.\lambda f.\, f\, x :: 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b$
- $\lambda f.\lambda x.\, f\, x :: ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
- $\lambda f.\lambda x.\, f\,(x+1)\,(x+1) :: (nat \Rightarrow nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$

$$(\lambda f.\lambda x.\, f\,(x+1)\,(x+1))\ \texttt{add}\ 20$$
$$\twoheadrightarrow_\beta (\lambda x.\, \texttt{add}\,(x+1)\,(x+1))\ 20$$
$$\twoheadrightarrow_\beta \texttt{add}\,(20+1)\,(20+1)$$
$$= (\lambda x.\lambda y.\, x+y)\,(20+1)\,(20+1)$$
$$\twoheadrightarrow_\beta (20+1)+(20+1)$$
$$= 42$$

A word about higher-order functions (II)

Exercise 3 (In Isabelle/HOL)

1. *Define a function* triple *which applies three times a given function to an argument*
2. *Using* triple*, apply three times the function* incr *on* 0
3. *Using* triple*, apply three times the function* app1 *on* [2,3]
4. *Using* map $:: ('a \Rightarrow 'b) \Rightarrow 'a\ list \Rightarrow 'b\ list$ *from the list* $[1, 2, 3]$ *build the list* $[2, 3, 4]$

Interlude: a word about semantics and verification

- To verify programs, formal reasoning on their semantics is crucial!
- To prove a property ϕ on a program P we need to precisely and exactly understand P's behavior

For many languages the semantics is given by the compiler (version)!

- C, Flash/ActionScript, JavaScript, Python, Ruby, . . .

Some languages have a (written) formal semantics:

- Java [a], subsets of C (hundreds of pages)
- Proofs are hard because of semantics complexity (*e.g.* KeY for Java)

[a] http://docs.oracle.com/javase/specs/jls/se7/html/index.html

Some have a small formal semantics:

- Functional languages: Haskell, subsets of (OCaml, F# and Scala)
- Proofs are easier since semantics essentially consists of a single rule

Constructor terms

Isabelle distinguishes between constructor and function symbols

- A function symbol is associated to a function, *e.g.* `inc`
- A constructor symbol is not associated to any function

Definition 10 (Constructor term)

A term containing only constructor symbols is a constructor term

A constructor term does not contain function symbols

Constructor terms (II)

All data are built using constructor terms **without** variables

...even if the representation is generally hidden by Isabelle/HOL

Example 11

- Natural numbers of type `nat` are terms: 0, (*Suc* 0), (*Suc* (*Suc* 0)), ...
- Integer numbers of type `int` are couples of natural numbers: $\ldots(0,2), (0,1), (0,0), (1,0), \ldots$

  where $(0,2) = (1,3) = (2,4) = \ldots$ all represent the *same* integer -2
- Lists are built using the operators
  - *Nil*: the empty list
  - *Cons*: the operator adding an element to the (head) of the list

    Be careful! the type of *Cons* is *Cons*`::'a ⇒ 'a list ⇒ 'a list`

The term *Cons* 0 (*Cons* (*Suc* 0) *Nil*) represents the list $[0,1]$

Constructor terms – the quiz

Quiz 2

Nil is a term?	V	True	R	False
Nil is a constructor term?	V	True	R	False
(*Cons* (*Suc* 0) *Nil*) is a constructor term?	V	True	R	False
((*Suc* 0), *Nil*) is a constructor term?	V	True	R	False
(*inc* (*Suc* 0)) is a constructor term?	V	True	R	False
(*Cons* *x* *Nil*) is a constructor term?	V	True	R	False
(*inc* *x*) is a constructor term?	V	True	R	False

Constructor terms: Isabelle/HOL

For most of constructor terms there exists shortcuts:

- Usual decimal representation for naturals, integers and rationals 1, 2, -3, -45.67676, ...
- [] and # for lists, e.g. *Cons* 0 (*Cons* (*Suc* 0) *Nil*) = 0#(1#[]) = [0, 1] (similar to [] and :: of OCaml)
- Strings using 2 quotes e.g. `''toto''` (instead of `"toto"`)

Exercise 4

1. *Prove that 3 is equivalent to its constructor representation*
2. *Prove that* $[1,1,1]$ *is equivalent to its constructor representation*
3. *Prove that the first element of list* $[1,2]$ *is* 1
4. *Infer the constructor representation of rational numbers of type* `rat`
5. *Infer the constructor representation of strings*

Isabelle Theory Library

Isabelle comes with a huge library of useful theories

- Numbers: Naturals, Integers, Rationals, Floats, Reals, Complex ...
- Data structures: Lists, Sets, Tuples, Records, Maps ...
- Mathematical tools: Probabilities, Lattices, Random numbers, ...

All those theories include types, functions and lemmas/theorems

Example 12

Let's have a look to a simple one `Lists.thy`:

- Definition of the datatype (with shortcuts)
- Definitions of functions (e.g. `append`)
- Definitions and proofs of lemmas (e.g. `length_append`)
  lemma "length (xs @ ys) = length xs + length ys"
- Exportation rules for SML, Haskell, Ocaml, Scala (`code_printing`)

Isabelle Theory Library: using functions on lists

Some functions of `Lists.thy`

- `append:: 'a list ⇒ 'a list ⇒ 'a list`
- `rev:: 'a list ⇒ 'a list`
- `length:: 'a list ⇒ nat`
- `map:: ('a ⇒ 'b) ⇒ 'a list ⇒ 'b list`

Exercise 5

1. *Apply the* `rev` *function to list* $[1, 2, 3]$
2. *Prove that for all value* x*, reverse of the list* $[x]$ *is equal to* $[x]$
3. *Prove that* `append` *is associative*
4. *Prove that* `append` *is not commutative*
5. *Using* `map`*, from the list* $[1, 2, 3]$ *build the list* $[2, 4, 6]$
6. *Prove that* `map` *does not change the size of a list*

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   - Logic everywhere!
   - Function evaluation using term rewriting
   - Partial functions

Defining functions using equations

- Defining functions using λ-terms is hardly usable for programming
- Isabelle/HOL has a "`fun`" operator as other functional languages

Definition 13 (`fun` operator for defining (recursive) functions)

`fun` f :: "$\tau_1 \Rightarrow \ldots \Rightarrow \tau_n \Rightarrow \tau$"
`where`
" $f\ t_1^1 \ldots t_n^1 \quad = \quad r^1$ " |
...
" $f\ t_1^m \ldots t_n^m \quad = \quad r^m$ "

where for all $i = 1 \ldots n$ and $k = 1 \ldots m$ $(t_i^k :: \tau_i)$ are constructor terms **possibly with** variables, and $(r^k :: \tau)$

Example 14 (The `member` function on lists (2 versions in `cm2.thy`))

```
fun member:: "'a => 'a list => bool"
where
"member e []      = False" |
"member e (x#xs) = (if e=x then True else (member e xs))"
```

Function definition – the quiz

Quiz 3 (Is this function definition correct? V Yes | R No)

```
fun f:: "nat ⇒ nat ⇒ bool"
where
"f x y = (x + y)"
```

Quiz 4 (Is this function definition correct? V Yes | R No)

```
fun g:: "nat ⇒ nat ⇒ bool"
where
"g 0 y = False"
```

Quiz 5 (Is this function definition correct? V Yes | R No)

```
fun pos:: "nat ⇒ bool"
where
"pos 0 = False" |
"pos (Suc x) = True"
```


Function definition – the quiz (II)

Quiz 6 (Is this function definition correct? V Yes | R No)

```
fun pos2:: "nat ⇒ bool"
where
"pos2 0 = False" |
"pos2 (x + 1) = True"
```

Quiz 7 (Is this function definition correct? V Yes | R No)

```
fun isDivisor:: "nat ⇒
nat ⇒ bool"
where
"isDivisor x y = (∃ z. x * z = y)"
```


Total and partial Isabelle/HOL functions

Definition 15 (Total and partial functions)

A function is *total* if it has a value (a result) for all elements of its domain. A function is *partial* if it is not total.

Definition 16 (Complete Isabelle/HOL function definition)

fun f :: "$\tau_1 \Rightarrow \ldots \Rightarrow \tau_n \Rightarrow \tau$"
where
" $f\ t_1^1 \ldots t_n^1 \;=\; r^1$ " |
...
" $f\ t_1^m \ldots t_n^m \;=\; r^m$ "

f is *complete* if any call $f\ t_1 \ldots t_n$ with $(t_i :: \tau_i)$, $i = 1 \ldots n$ is covered by one case of the definition.

Example 17 (Isabelle/HOL "`Missing patterns`" warning)

When the definition of f is not complete, an uncovered call of f is shown.

Total and partial Isabelle/HOL functions (II)

Theorem 18

Complete and terminating Isabelle/HOL functions are total, otherwise they are partial.

Question 1

Why termination of f is necessary for f to be total?

Remark 1

All functions in Isabelle/HOL needs to be terminating!

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Logic everywhere!

In the end, everything is defined using logic:

- data, data structures: constructor terms
- properties: lemmas (logical formulas)
- programs: functions (also logical formulas!)

Definition 19 (Equations (or simplification rules) defining a function)

A function `f` consists of a set of `f.simps` of equations on terms.

To visualize a lemma/theorem/simplification rule .... `thm`

For instance: `thm "length_append"`, `thm "append.simps"`

To *find* the name of a lemma, etc. .... `find_theorems`

For instance: `find_theorems "append" "_ + _"`

Exercise 6

Use Isabelle/HOL to find the following formulas:

- *definition of* `member` *(we just defined) and of* `nth` *(part of List.thy)*
- *find the lemma relating* `rev` *(part of List.thy) and* `length`

Evaluating functions by rewriting terms using equations

The append function (aliased to `@`) is defined by the 2 equations:

```
(1) append    Nil  x = x                  (* recall that Nil=[] *)
(2) append (x#xs)  y = (x#(append xs y))
```

Replacement of equals by equals = Term rewriting

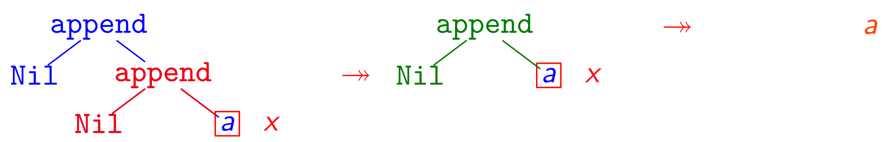
The first equation `(append Nil x) = x` means that

- *(concatenating the empty list with any list x)* is **equal** to *x*
- we can thus replace
  - any term of the form `(append Nil t)` by `t` (for any value `t`)
  - wherever and whenever we encounter such a term `append Nil t`

Term Rewriting in three slides

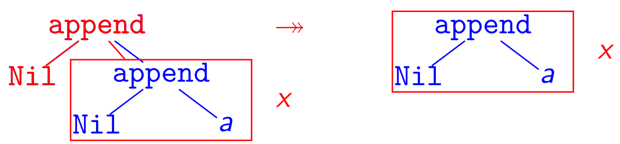
- Rewriting term `(append [] (append [] a))` using

```
(1) append    Nil  x  =  x
(2) append (x#xs)  y  =  (x#(append xs y))
```



- We note `(append Nil (append Nil a))` $\twoheadrightarrow$ `(append Nil a)` if
  - there exists a position in the term where the rule matches
  - there exists a substitution $\sigma : \mathcal{X} \mapsto \mathcal{T}(\mathcal{F})$ for the rule to match. On the example $\sigma = \{x \mapsto a\}$
- We also have `(append Nil a)` $\twoheadrightarrow$ `a`

and



Term Rewriting in three slides – Formal definitions

Definition 20 (Substitution)

A substitution σ is a function replacing variables of $\mathcal{X}$ by terms of $\mathcal{T}(\mathcal{F}, \mathcal{X})$ in a term of $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Example 21

Let $\mathcal{F} = \{f : 3, h : 1, g : 1, a : 0\}$ and $\mathcal{X} = \{x, y, z\}$.

Let σ be the substitution $\sigma = \{x \mapsto g(a), y \mapsto h(z)\}$.

Let $t = f(h(x), x, g(y))$.

We have $\sigma(t) = f(h(g(a)), g(a), g(h(z)))$.

Term Rewriting in three slides – Formal definitions (II)

Definition 22 (Rewriting using an equation)

A term s can be *rewritten* into the term t (denoted by $s \twoheadrightarrow t$) using an Isabelle/HOL equation `l=r` if there exists a subterm u of s and a substitution σ such that $u = \sigma(\texttt{l})$. Then, t is the term s where subterm u has been replaced by $\sigma(\texttt{r})$.

Example 23

Let $s = f(g(a), c)$ and `g(x) = h(g(x),b)` the Isabelle/HOL equation.

we have $f(\; g(a) \;, c) \twoheadrightarrow f(\; h(g(a), b) \;, c)$
because `g(x)` = `h(g(x),b)` and $\sigma = \{\texttt{x} \mapsto a\}$

On the opposite $t = f(a, c)$ cannot be rewritten by `g(x) = h(g(x),b)`.

Remark 2

Isabelle/HOL rewrites terms using equations in the order of the function definition and only from left to right.

Term rewriting – the quiz

Quiz 8

Let $\mathcal{F} = \{f : 2, g : 1, a : 0\}$ and $\mathcal{X} = \{x, y\}$.

- *Rewriting the term $f(g(g(a)))$ with equation $g(x) = x$ is*
  V *Possible* || R *Impossible*
- *To rewrite the term $f(g(g(a)))$ with $g(x) = x$ the substitution σ is*
  V $\{x \mapsto a\}$ || R $\{x \mapsto g(a)\}$
- *Rewriting the term $f(g(g(y)))$ with equation $g(x) = x$ is*
  V *Possible* || R *Impossible*
- *Rewriting the term $f(g(g(y)))$ with equation $g(f(x)) = x$ is*
  V *Possible* || R *Impossible*

Isabelle evaluation = rewriting terms using equations

```
(1) append    Nil   x  =  x
(2) append (x#xs)   y  =  (x#(append xs y))
```

Rewriting the term: `append [1,2] [3,4]` with (1) then (2) (Rmk 2)

First, recall that `[1,2] = (1#(2#Nil))` and `[3,4] = (3#(4#Nil))`!

`append (1#(2#Nil)) (3#(4#Nil))`
`(1# (append (2#Nil) (3#(4#Nil))))}`
with $\sigma = \{x \mapsto 1, xs \mapsto (2\#Nil), y \mapsto (3\#(4\#Nil))\}$
	$\not\twoheadrightarrow_{(1)}$ $\twoheadrightarrow_{(2)}$

`(1# (append (2#Nil) (3#(4#Nil))))`
`(1# (2#(append Nil (3#(4#Nil)))))`
with $\sigma = \{x \mapsto 2, xs \mapsto Nil, y \mapsto (3\#(4\#Nil))\}$
	$\twoheadrightarrow_{(2)}$

`(1#(2# (append Nil (3#(4#Nil)))))`
`(1#(2# (3#(4#Nil)) )) = [1,2,3,4] !`
with $\sigma = \{x \mapsto (3\#(4\#Nil))\}$
	$\twoheadrightarrow_{(1)}$

Example 24

See demo of step by step rewriting in Isabelle/HOL!

Isabelle evaluation = rewriting terms using equations (II)

```
(1) member e []         = False
(2) member e (x # xs)= (if e=x then True else (member e xs))
```

Evaluation of test: `member 2 [1,2,3]`

↠ `if 2=1 then True else (member 2 [2,3])`
  by equation (2), because `[1,2,3] = 1#[2,3]`
↠ `if False then True else (member 2 [2,3])`
  by Isabelle equations defining equality on naturals
↠ `member 2 [2,3]`
  by Isabelle equation `(if False then x else y = y)`
↠ `if 2=2 then True else (member 2 [3])`
  by equation (2), because `[2,3] = 2#[3]`
↠ `if True then True else (member 2 [3])`
  by Isabelle equations defining equality on naturals
↠ `True`
  by Isabelle equation `(if True then x else y = x)`

Lemma simplification= Rewriting + Logical deduction

```
(1) member e []         = False
(2) member e (x # xs)= (if e=x then True else (member e xs))
```

Proving the lemma: `member y [z,y,v]`

↠ `if y=z then True else (member y [y,v])`
  by equation (2), because `[z,y,v] = z#[y,v]`
↠ `if y=z then True else (if y=y then True else (member y [v]))`
  by equation (2), because `[y,v] = y#[v]`
↠ `if y=z then True else (if True then True else (member y [v]))`
  because `y=y` is trivially True
↠ `if y=z then True else True`
  by Isabelle equation `(if True then x else y = x)`
↠ `True`
  by logical deduction `(if b then True else True)`⟷`True`

Lemma simplification= Rewriting + Logical deduction (II)

```
(1) member e []         = False
(2) member e (x # xs)   = (if e=x then True else (member e xs))

(3) append [] x         = x
(4) append (x # xs) y   = x # (append xs y)
```

Exercise 7

Is it possible to prove the lemma `member u (append [u] v)` *by simplification/rewriting?*

Exercise 8

Is it possible to prove the lemma `member v (append u [v])` *by simplification/rewriting?*

Demo of rewriting in Isabelle/HOL!

Evaluation of partial functions

Evaluation of partial functions using rewriting by equational definitions may not result in a constructor term

Exercise 9

Let `index` *be the function defined by:*

```
fun index:: "'a => 'a list => nat"
where
"index y (x#xs) = (if x=y then 0 else 1+(index y xs))"
```

- *Define the function in Isabelle/HOL*
- *What does it computes?*
- *Why is* `index` *a partial function? (What does Isabelle/HOL says?)*
- *For* `index`*, give an example of a call whose result is:*
  - *a constructor term*
  - *a match failure*
- *Define the property relating functions* `index` *and* `List.nth`

Scala export + Demo

To export functions to Haskell, SML, Ocaml, Scala .... export_code

For instance, to export the member and index functions to Scala:

```
export_code member index in Scala
```

test.scala

```
object cm2 {
  def member[A : HOL.equal](e: A, x1: List[A]): Boolean =
  (e, x1) match {
    case (e, Nil) => false
    case (e, x :: xs) => (if (HOL.eq[A](e, x)) true
                          else member[A](e, xs))
  }
  def index[A : HOL.equal](y: A, x1: List[A]): Nat =
  (y, x1) match {
    case (y, x :: xs) =>
      (if (HOL.eq[A](x, y)) Nat(0)
       else Nat(1) + index[A](y, xs))
  }
}
```