Analyse et Conception Formelles

Lesson 1

–

Propositional logic
First order logic

Bibliography

- *Cours de logique pour l'informatique*, J-F. Raskin, `http://www.ulb.ac.be/di/ssd/jfr/info-148.html`
- *Logique et fondements de l'informatique* de Richard Lassaigne et Michel de Rougemont. Hermes 1993.

A selected bibliography on the Isabelle/HOL prover

- `http://people.irisa.fr/Thomas.Genet/ACF/BiblioIsabelle`

The web page of the course

- `http://people.irisa.fr/Thomas.Genet/ACF`

Solutions of Isabelle/HOL exercises (uploaded after each lecture)

- `http://people.irisa.fr/Thomas.Genet/ACFSol`

Acknowledgements

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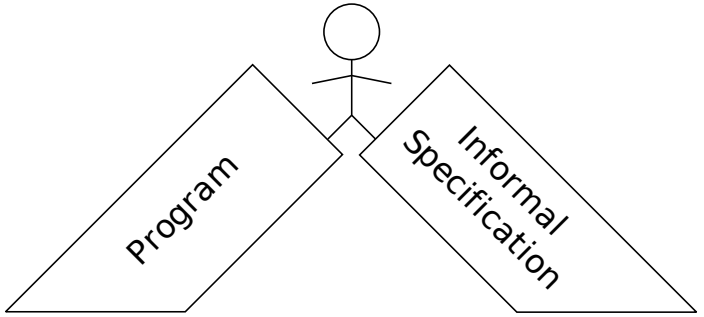
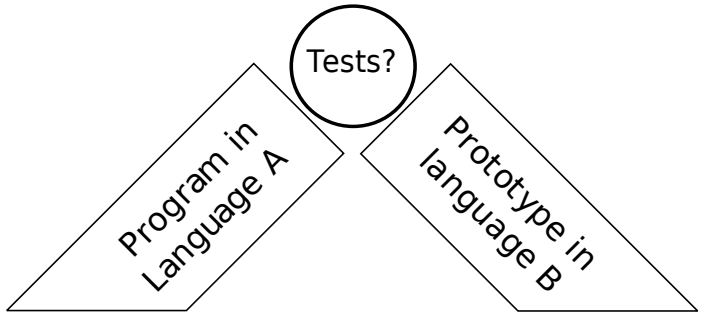
Outline

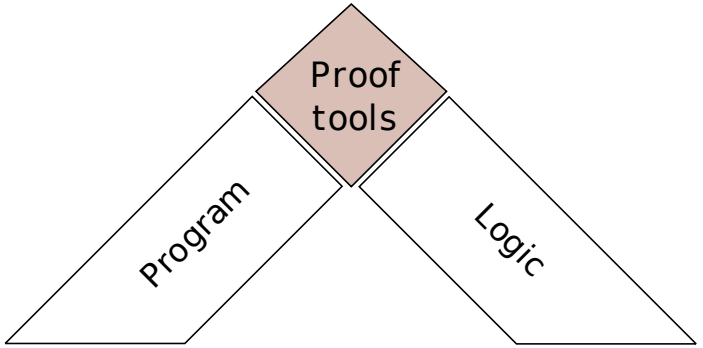
Why using logic for specifying/verifying programs?



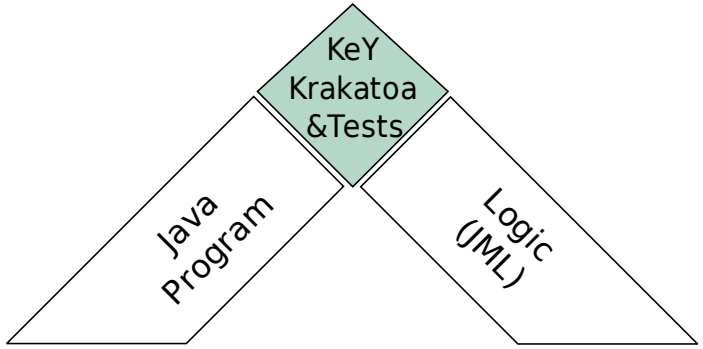
Why using logic for specifying/verifying programs?



Why using logic for specifying/verifying programs?



Why using functional paradigm to program?



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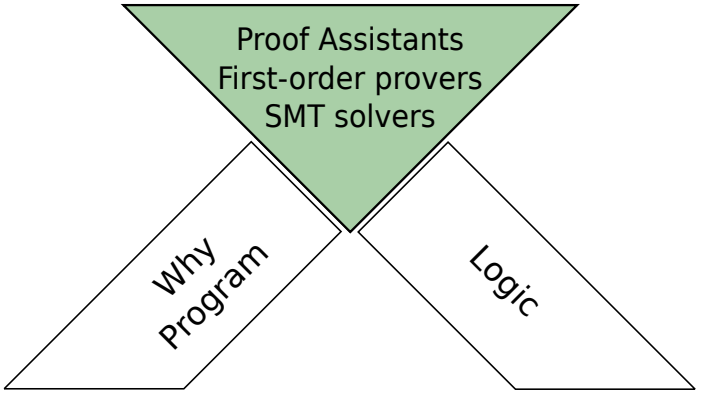

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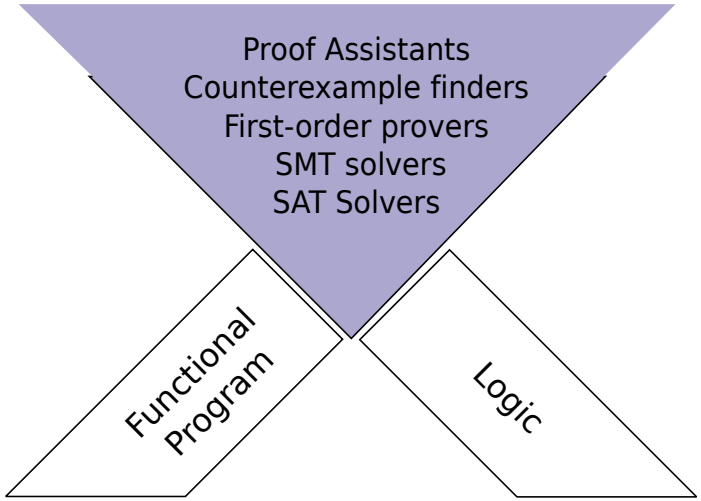

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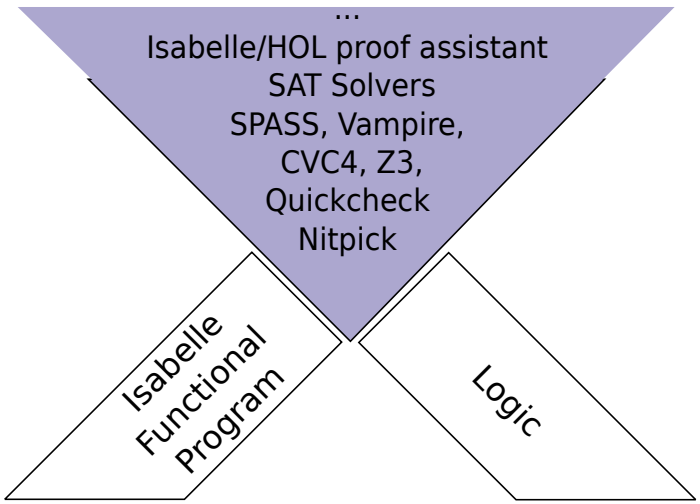

Propositional logic: syntax and interpretations

Definition 1 (Propositional formula)

Let P be a set of propositional variables. The set of propositional formula is defined by

$\phi ::= p \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \phi_1 \longrightarrow \phi_2$ where $p \in P$

Definition 2 (Propositional interpretation)

An *interpretation* I associates to variables of P a value in $\{\texttt{True}, \texttt{False}\}$.

Example 3

Let $\phi = (p_1 \wedge p_2) \longrightarrow p_3$. Let I be the interpretation such that $I[\![p_1]\!] = \texttt{True}$, $I[\![p_2]\!] = \texttt{True}$ and $I[\![p_3]\!] = \texttt{False}$.

Propositional logic: syntax and interpretations (II)

We extend the domain of I to formulas as follows:

$$I[\![\neg\phi]\!] = \begin{cases} \texttt{True} \text{ iff } I[\![\phi]\!] = \texttt{False} \\ \texttt{False} \text{ iff } I[\![\phi]\!] = \texttt{True} \end{cases}$$

$I[\![\phi_1 \vee \phi_2]\!] = \texttt{True}$ iff $I[\![\phi_1]\!] = \texttt{True}$ or $I[\![\phi_2]\!] = \texttt{True}$

$I[\![\phi_1 \wedge \phi_2]\!] = \texttt{True}$ iff $I[\![\phi_1]\!] = \texttt{True}$ and $I[\![\phi_2]\!] = \texttt{True}$

$$I[\![\phi_1 \longrightarrow \phi_2]\!] = \texttt{True} \text{ iff } \begin{cases} I[\![\phi_1]\!] = \texttt{False} \text{ or} \\ I[\![\phi_1]\!] = \texttt{True} \text{ and } I[\![\phi_2]\!] = \texttt{True} \end{cases}$$

Example 4

Let $\phi = (p_1 \wedge p_2) \longrightarrow p_3$ and I the interpretation such that $I[\![p_1]\!] = \texttt{True}$, $I[\![p_2]\!] = \texttt{True}$ and $I[\![p_3]\!] = \texttt{False}$.

We have $I[\![p_1 \wedge p_2]\!] = \texttt{True}$ and $I[\![(p_1 \wedge p_2) \longrightarrow p_3]\!] = \texttt{False}$.

Propositional logic: syntax and interpretations (III)

The presentation using truth tables is generally preferred:

a	$\neg a$
False	True
True	False

a	b	$a \vee b$
False	False	False
True	False	True
False	True	True
True	True	True

a	b	$a \wedge b$
False	False	False
True	False	False
False	True	False
True	True	True

a	b	$a \longrightarrow b$
False	False	True
True	False	False
False	True	True
True	True	True

Propositional logic: models

Definition 5 (Propositional model)

I is a *model* of ϕ, denoted by $I \models \phi$, if $I[\![\phi]\!] = \texttt{True}$.

Definition 6 (Valid formula/Tautology)

A formula ϕ is *valid*, denoted by $\models \phi$, if for all I we have $I \models \phi$.

Example 7

Let $\phi = (p_1 \wedge p_2) \longrightarrow p_3$ and $\phi' = (p_1 \wedge p_2) \longrightarrow p_1$. Let I be the interpretation such that $I[\![p_1]\!] = \texttt{True}$, $I[\![p_2]\!] = \texttt{True}$ and $I[\![p_3]\!] = \texttt{False}$. We have $I \not\models \phi$, $I \models \phi'$, and $\models \phi'$.

Propositional logic: decidability and tools in Isabelle/HOL

Property 1

In propositional logic, given ϕ, the following problems are decidable:

- *Is $\models \phi$?*
- *Is there an interpretation I such that $I \models \phi$?*
- *Is there an interpretation I such that $I \not\models \phi$?*

- To automatically prove that $\models \phi$ ..... `apply auto`
  (if the formula is not valid, there remains some unsolved goals)
- To build I such that $I \not\models \phi$ (or $I \models \neg\phi$) ..... `nitpick`
  (i.e. find a counterexample... may take some time on large formula)

Other useful commands

- To close the proof of a proven formula ..... `done`
- To abandon the proof of an unprovable formula ..... `oops`
- To abandon the proof of (potentially) provable formula ..... `sorry`

Writing and proving propositional formulas in Isabelle/HOL

Example 8 (Valid formula)

```
lemma "(p1 /\ p2) --> p1"
apply auto
done
```

Example 9 (Unprovable formula)

```
lemma "(p1 /\ p2) --> p3"
nitpick
oops
```

Isabelle/HOL: ASCII notations

Symbol	ASCII notation
`True`	`True`
`False`	`False`
$\wedge$	`/\`
$\vee$	`\/`
$\neg$	`~`
$\neq$	`~=`
$\longrightarrow$	`-->`
$\longleftrightarrow$	`=`
$\forall$	`ALL`
$\exists$	`?`
λ	`%`

Propositional logic: exercises in Isabelle/HOL

Exercise 1

Using Isabelle/HOL, for each formula, say if it is valid or give a counterexample interpretation, otherwise.

1. $A \vee B$
2. $(((A \wedge B) \longrightarrow \neg C) \vee (A \longrightarrow B)) \longrightarrow A \longrightarrow C$
3. *If it rains, Robert takes his umbrella. Robert does not have his umbrella hence it does not rain.*
4. $(A \longrightarrow B) \longleftrightarrow (\neg A \vee B)$

First-order logic (FOL) / Predicate logic

1. Terms and Formulas
2. Interpretations
3. Models
4. Logic consequence and verification

First-order logic: terms

Definition 10 (Terms)

Let $\mathcal{F}$ be a set of symbols and ar a function such that $ar : \mathcal{F} \Rightarrow \mathbb{N}$ associating each symbol of $\mathcal{F}$ to its arity (the number of parameter). Let $\mathcal{X}$ be a variable set.

The set $\mathcal{T}(\mathcal{F}, \mathcal{X})$, the set of *terms* built on $\mathcal{F}$ and $\mathcal{X}$, is defined by:
$\mathcal{T}(\mathcal{F}, \mathcal{X}) = \mathcal{X} \cup \{f(t_1, \ldots, t_n) \mid ar(f) = n \text{ and } t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}$

Example 11

Let $\mathcal{F} = \{f : 1, g : 2, a : 0, b : 0\}$ and $\mathcal{X} = \{x, y, z\}$.

$f(x)$, a, z, $g(g(a, x), f(a))$, $g(x, x)$ are terms and belong to $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

f, $a(b)$, $f(a, b)$, $x(a)$, $f(a, f(b))$ do not belong to $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

First-order logic: formula syntax

Definition 12 (Formulas)

Let P be a set of predicate symbols all having an arity, i.e. $ar : P \Rightarrow \mathbb{N}$.
The set of formulas defined on $\mathcal{F}, \mathcal{X}$ and P is:

$$\phi ::= \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \phi_1 \longrightarrow \phi_2 \mid \forall x.\phi \mid \exists x.\phi \mid p(t_1, \ldots, t_n)$$

where $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $x \in \mathcal{X}$, $p \in P$ and $ar(p) = n$.

Example 13

Let $P = \{p : 1, q : 2, \leq : 2\}$, $\mathcal{F} = \{f : 1, g : 2, a : 0\}$ and $\mathcal{X} = \{x, y, z\}$.
The following expressions are all formulas:

- $p(f(a))$
- $q(g(f(a), x), y)$
- $\forall x.\exists y.y \leq x$
- $\forall x.\forall y.\forall z.x \leq y \wedge y \leq z \longrightarrow x \leq z$

First-order logic syntax: the quiz

Quiz 1

Let $P = \{p : 1, q : 2, \leq : 2\}$, $\mathcal{F} = \{f : 1, g : 2, a : 0\}$ *and* $\mathcal{X} = \{x, y, z\}$.

- $f(g(a))$ *is a term* V True R False
- a *is a term* V True R False
- x *is a term* V True R False
- $\forall x.\, x$ *is a term* V True R False
- $\forall x.\, x$ *is a formula* V True R False
- $p(f(g(a, x)))$ *is a formula* V True R False
- $\forall x.\, p(x) \wedge x \leq y$ *is a formula* V True R False

Interlude: a touch of lambda-calculus

We need to define *anonymous* functions

- Classical notation for functions

  $f : \mathbb{N} \times \mathbb{N} \Rightarrow \mathbb{N}$ or, for short, $f : \mathbb{N}^2 \Rightarrow \mathbb{N}$
  $f(x, y) = x + y$ $f(x, y) = x + y$

- Lambda-notation of functions

  $f : \mathbb{N}^2 \Rightarrow \mathbb{N}$
  $f = \lambda(x, y).\, x + y$

$\lambda x\, y.\, x + y$ is an anonymous function adding two naturals

This corresponds to

- `fun x y -> x+y` in OCaml/Why3
- `(x: Int, y:Int) => x + y` in Scala

Interlude: a touch of lambda-calculus (in Isabelle HOL)

Isabelle/HOL also use function update using (:=) as in:

- $(\lambda x.x)(0 := 1, 1 := 2)$ the identity function except for 0 that is mapped to 1 and 1 that is mapped to 2
- $(\lambda x.\_)(a := b)$ a function taking one parameter and whose result is unspecified except for value a that is mapped to b

Predicates in Isabelle/HOL

- A predicate is a function mapping values to $\{\texttt{True}, \texttt{False}\}$

  For instance the predicate p on $\{a, b\}$
  $p = (\lambda x.\_)(a := \mathit{False}, b := \mathit{False})$

First-order formulas: interpretations and valuations

Definition 14 (First-order interpretation)

Let ϕ be a formula and D a domain. An *interpretation* I of ϕ on the domain D associates:

- a function $f_I : D^n \Rightarrow D$ to each symbol $f \in \mathcal{F}$ such that $ar(f) = n$,
- a function $p_I : D^n \Rightarrow \{\texttt{True}, \texttt{False}\}$ to each predicate symbol $p \in P$ such that $ar(p) = n$.

Example 15 (Some interpretations of $\phi = \forall x.ev(x) \longrightarrow od(s(x))$)

- Let I be the interpretation such that domain $D = \mathbb{N}$ and $s_I \equiv \lambda x.x + 1$ $ev_I \equiv \lambda x.((x \bmod 2) = 0)$ $od_I \equiv \lambda x.((x \bmod 2) = 1)$
- Let I' be the interpretation such that domain $D = \{a, b\}$ and $s_{I'} \equiv \lambda x.\mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ a$ $ev_{I'} \equiv \lambda x.(x = a)$ $od_{I'} \equiv \lambda x.\mathit{False}$

Definition 16 (Valuation)

Let D be a domain. A *valuation* V is a function $V : \mathcal{X} \Rightarrow D$.

First-order logic: interpretations and valuations (II)

Definition 17

The interpretation I of a formula ϕ for a valuation V is defined by:

- $(I, V)[\![x]\!] = V(x)$ if $x \in \mathcal{X}$
- $(I, V)[\![f(t_1, \ldots, t_n)]\!] = f_I((I, V)[\![t_1]\!], \ldots, (I, V)[\![t_n]\!])$ if $f \in \mathcal{F}$ and $ar(f) = n$
- $(I, V)[\![p(t_1, \ldots, t_n)]\!] = p_I((I, V)[\![t_1]\!], \ldots, (I, V)[\![t_n]\!])$ if $p \in P$ and $ar(p) = n$
- $(I, V)[\![\phi_1 \vee \phi_2]\!] = \texttt{True}$ iff $(I, V)[\![\phi_1]\!] = \texttt{True}$ or $(I, V)[\![\phi_2]\!] = \texttt{True}$
- etc...
- $(I, V)[\![\forall x.\phi]\!] = \bigwedge_{d \in D} (I, V + \{x \mapsto d\})[\![\phi]\!]$
- $(I, V)[\![\exists x.\phi]\!] = \bigvee_{d \in D} (I, V + \{x \mapsto d\})[\![\phi]\!]$

where $(V + \{x \mapsto d\})(x) = d$ and $(V + \{x \mapsto d\})(y) = V(y)$ if $x \neq y$.

First-order logic: satisfiability, models, tautologies

Definition 18 (Satisfiability)

I and V *satisfy* ϕ (denoted by $(I, V) \models \phi$) if $(I, V)[\![\phi]\!] = \texttt{True}$.

Definition 19 (First-order Model)

An interpretation I is a *model* of ϕ, denoted by $I \models \phi$, if for all valuation V we have $(I, V) \models \phi$.

Definition 20 (First-order Tautology)

A formula ϕ is a tautology if all its interpretations are models, i.e. $(I, V) \models \phi$ for all interpretations I and all valuations V.

Remark 1

Free variables are universally quantified (e.g. $P(x)$ equivalent to $\forall x.\ P(x)$)

First-order logic: decidability and tools in Isabelle/HOL

Property 2

In first-order logic, given ϕ, the following problems are undecidable:

- *Is $\models \phi$?*
- *Is there an interpretation I (and valuation V) such that $(I, V) \models \phi$?*
- *Is there an interpretation I (and valuation V) such that $(I, V) \not\models \phi$?*

- Try to automatically prove $\models \phi$ ..... `apply auto`
  Uses decision procedures (*e.g.* arithmetic) to try to prove the formula.
  If it does not succeed, it does not mean that the formula is unprovable!
- Try to build I and V such that $(I, V) \not\models \phi$ ..... `nitpick`
  If it does not succeed, it does not mean that there is no counterexample!

First-order logic: exercises in Isabelle/HOL

Exercise 2

Using Isabelle/HOL, for each formula, say if it is valid or give a counterexample interpretation and valuation otherwise.

1. $\forall x.\ p(x) \longrightarrow \exists x.p(x)$
2. $\exists x.\ p(x) \longrightarrow \forall x.p(x)$
3. $\forall x.\ ev(x) \longrightarrow od(s(x))$
4. $\forall x\ y.\ x > y \longrightarrow x + 1 > y + 1$
5. $x > y \longrightarrow x + 1 > y + 1$
6. $\forall m\ n.\ (\neg(m < n) \wedge m < n + 1) \longrightarrow m = n$
7. $\forall x.\ \exists y.\ x + y = 0$
8. $\forall y.\ (\neg p(f(y))) \longleftrightarrow p(f(y))$
9. $\forall y.\ (p(f(y)) \longrightarrow p(f(y + 1)))$

Isabelle/HOL notations: priority, associativity, shorthands

- Here are the logical operators in decreasing order of priority:
  - $=, \neg, \wedge, \vee, \longrightarrow, \forall, \exists$
  - «a prioritary operator first chooses its operands»
- For instance
  - $\neg\neg P = P$ means $\neg\neg(P = P)$!
  - $A \wedge B = B \wedge A$ means $A \wedge (B = B) \wedge A$!
  - $P \wedge \forall x.Q(x)$ will be parsed as $(P \wedge \forall)x.Q(x)$!

    Hence, write $P \wedge (\forall x.Q(x))$ instead!
- All binary operators are associative to the right, for instance $A \longrightarrow B \longrightarrow C$ is equivalent to $A \longrightarrow (B \longrightarrow C)$
- Nested quantifications $\forall x.\ \forall y.\ \forall z.\ P$ can be abbreviated into $\forall x\ y\ z.\ P$
- Free variables are universally quantified, i.e. $P(x)$ is equiv. to $\forall x.\ P(x)$

  All Isabelle/HOL tools will prefer $P(x)$ to $\forall x.\ P(x)$

First-order logic: satisfiability and models

Definition 21 (Satisfiable formula)

A formula ϕ is *satisfiable* if there exists an interpretation I and a valuation V such that $(I, V) \models \phi$.

Example 22

Let $\phi = p(f(y))$ with $\mathcal{F} = \{f : 1\}$, $P = \{p : 1\}$, $\mathcal{X} = \{y\}$.
The formula ϕ is satisfiable (there exists (I, V) such that $(I, V) \models \phi$)

- Let I be the interp. s.t. $D = \{0, 1\}$, $p_I \equiv \lambda x.(x = 0)$, $f_I = \lambda x.x$
- Let V be the valuation such that $V(y) = 0$

We have $(I, V) \models \phi$. With $V'(y) = 1$, $(I, V') \not\models \phi$. Hence, I is not a model of ϕ.

- Let I' be the interp. s.t. $D = \{0, 1\}$, $p_{I'} \equiv \lambda x.(x = 0)$, $f_{I'} = \lambda x.0$

We have $(I', V) \models \phi$ for all valuation V, hence I' is a model of ϕ.

Satisfiability – the quiz

Quiz 2

Let $P = \{p : 1\}$, $\mathcal{F} = \{f : 1, a : 0, b : 0\}$ *and* $\mathcal{X} = \{x\}$.

- $f(a)$ *is satisfiable* V True R False
- $p(f(a))$ *is satisfiable* V True R False
- $p(f(x))$ *is satisfiable* V True R False
- $p(f(x))$ *is a tautology* V True R False
- $\neg p(f(x))$ *is satisfiable* V True R False
- $\neg p(f(x)) \wedge p(f(x))$ *is satisfiable* V True R False
- $p(f(a)) \longrightarrow p(f(b))$ *is satisfiable* V True R False

First-order logic: contradictions

Definition 23 (Contradiction)

A formula is *contradictory* (or *unsatisfiable*) if it cannot be satisfied, i.e. $(I, V) \not\models \phi$ for all interpretation I and all valuation V.

Property 3

A formula ϕ is contradictory iff $\neg\phi$ is a tautology.

Example 24 (See in Isabelle `cm1.thy` file)

Let $\phi = (\forall y.\, \neg p(f(y))) \longleftrightarrow (\forall y.\, p(f(y)))$. The formula ϕ is contradictory and $\neg\phi$ is a tautology.