Asymptotic Normality in Partially Observed Diffusions with Small Noise: Application to FDI

This paper is dedicated to Tyrone E. Duncan on the occasion of his 60th birthday

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Abstract. The problem of residual evaluation for fault detection in partially observed diffusions is investigated, using the local asymptotic approach, under the small noise asymptotics. The score function (i.e. the gradient of the log–likelihood function) evaluated at the nominal value of the parameter, and suitably normalized, is used as residual. It is proved that this residual is asymptotically Gaussian, with mean zero under the null hypothesis, with a different mean (depending linearly on the parameter change) and the same covariance matrix under the contiguous alternative hypothesis. This result relies on the local asymptotic normality (LAN) property for the family of probability distributions of the observation process, which is also proved.

Keywords: fault detection, residual evaluation, local approach, asymptotic normality, LAN

1 Introduction

The problem of fault detection and isolation (FDI) in dynamical properties of signals and systems has received growing attention, and has been theoretically and experimentally investigated with different types of approaches. Among these, the local approach is known to be very powerful and has been successfully applied in many practical situations. Some early and fundamental works on the subject include Benveniste, Basseville and Moustakides [2], Benveniste, Métivier and Priouret [3], Basseville [1], etc. As explained there, the local approach consists of the following two steps. The first step, called residual generation, is to propose a statistics, called the residual, which depends only on the observations and on the nominal value of the parameter, and which ideally should be close to zero under the null hypothesis, and significantly different from zero under the alternative hypothesis. The second step, called residual evaluation, is to study the asymptotic behaviour of the residual under the null hypothesis and under a contiguous alternative hypothesis, and to design a simple test based on the residual.

In the case of completely observed stationary Markov processes (controlled by the parameter), Basseville [1], Delyon, Juditsky and Benveniste [7]

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have considered some residuals (such as the efficient score, the quasi–score, etc.) and have investigated their asymptotic behaviour when the size N of the sample goes to infinity : typically, the residual ζ_N satisfies the following central limit theorem as $N \uparrow \infty$

$$\zeta_N \Longrightarrow \begin{cases} \mathcal{N}(0, \Sigma(\alpha)) \ , & \text{under } \mathbb{P}_\alpha, \\ \\ \mathcal{N}(M(\alpha)\, \Delta, \Sigma(\alpha)) \ , & \text{under } \mathbb{P}_{\alpha+\Delta/\sqrt{N}}, \end{cases}$$

where α is the nominal value of the parameter. ¿From this result, one can design the optimum test by reducing the original problem of detecting a change in a dynamical system, to the simple and universal static problem of detecting a change in the mean of a Gaussian r.v., see [1].

In the case of partially observed diffusions, Cérou, LeGland and Newton [5] have addressed the problem of residual generation. They considered the score function, i.e. the gradient of the log-likelihood function w.r.t. the parameter, evaluated at the nominal value, as the residual, and proposed an efficient numerical approximation using particle filters. In this paper, the corresponding residual evaluation problem is studied. Instead of the large time asymptotics, the small noise asymptotics is used here, which usually is simpler to deal with, see Kutoyants [10].

The paper is organized as follows. Section 2 presents the model and gives the expression of the residual. Some preliminary computations based on small noise expansion, are presented in Section 3. Asymptotic normality of the residual under the nominal hypothesis is studied in Section 4. In Section 5, we prove the local asymptotic normality of the family of probability distributions of the observation process, and in Section 6 we prove the asymptotic normality of the residual under the contiguous alternative hypothesis, and we apply these results to the FDI problem.

2 Statistical Model and Residual Definition

On a measurable space $(\Omega, \mathcal{F}, P)$ are given

- for each $\varepsilon > 0$, a family $\mathcal{M}^\varepsilon = \{\mathbb{P}_{\theta,\varepsilon}, \theta \in \Theta\}$ of probability measures,
- two stochastic processes $X = \{X_t\,,\, 0 \le t \le T\}$ and $Y = \{Y_t\,,\, 0 \le t \le T\}$, taking values in $\mathbb{R}^m$ and $\mathbb{R}^d$ respectively,

such that under $\mathbb{P}_{\theta,\varepsilon}$

$$dX_t = b(\theta, X_t)dt + \varepsilon\, dW_t^\theta \ , \quad X_0 = \varepsilon\, \xi \ ,$$

$$dY_t = h(X_t)dt + \varepsilon\, dV_t^\theta \ , \quad Y_0 = 0 \ ,$$

where $\{W_t^\theta, 0 \leq t \leq T\}$ and $\{V_t^\theta, 0 \leq t \leq T\}$ are independent standard Wiener processes, and ξ is an $\mathbb{R}^m$–valued r.v. independent of the Wiener processes, which we assume to be a standard Gaussian r.v., without loss of generality.

Assume that Ω is the canonical space $C([0,T];\mathbb{R}^{m+d})$, in which case X and Y are the canonical processes on $C([0,T];\mathbb{R}^m)$ and $C([0,T];\mathbb{R}^d)$, respectively, and $\mathbb{P}_{\theta,\varepsilon}$ is the probability distribution of (X,Y). The set of parameters $\Theta \subset \mathbb{R}^p$ is compact, and the coefficients satisfy the following hypotheses

- the mapping $(\theta,x) \longmapsto b(\theta,x)$ is bounded, with bounded and Lipschitz continuous derivatives $\partial b(\theta,x)$ and $b'(\theta,x)$ (w.r.t. the parameter and the state variable respectively),
- the mapping $x \longmapsto h(x)$ is bounded, with bounded and Lipschitz continuous derivative $h'(x)$.

Our purpose is to design statistical tests to decide whether $\theta = \alpha$, corresponding to a nominal behaviour of the system, or $\theta \neq \alpha$, on the basis of a given observation path $\{Y_t, 0 \leq t \leq T\}$. Let $\mathcal{Y}_T$ denote the σ–algebra generated by the process Y. The hypotheses ensure that the probability measures in $\mathcal{M}^\varepsilon$ are mutually absolutely continuous, thus, according to Campillo and LeGland [4], the likelihood function for estimating the parameter θ given $\mathcal{Y}_T$ can be expressed as

$$L_\varepsilon(\theta) = \mathbb{E}^\dagger_{\theta,\varepsilon}[Z_T^\varepsilon \mid \mathcal{Y}_T] \ ,$$

where

$$Z_t^\varepsilon = \exp\{\frac{1}{\varepsilon^2} \int_0^t h^*(X_s)\, dY_s - \frac{1}{2\varepsilon^2} \int_0^t |h(X_s)|^2\, ds\} \ ,$$

for any $0 \leq t \leq T$, and where $\mathbb{P}^\dagger_{\theta,\varepsilon}$ is a probability measure on Ω, equivalent to $\mathbb{P}_{\theta,\varepsilon}$ with Radon–Nikodym derivative

$$\frac{d\mathbb{P}_{\theta,\varepsilon}}{d\mathbb{P}^\dagger_{\theta,\varepsilon}} = Z_T^\varepsilon \ ,$$

independent of $\theta \in \Theta$, so that under $\mathbb{P}^\dagger_{\theta,\varepsilon}$

$$dX_t = b(\theta, X_t)dt + \varepsilon\, dW_t^\theta, \qquad X_0 = \varepsilon\, \xi \ ,$$

and $\{Y_t, 0 \leq t \leq T\}$ is a Wiener process independent of $\{W_t^\theta, 0 \leq t \leq T\}$, with covariance matrix $\varepsilon^2 I_d$. Alternatively

$$L_\varepsilon(\theta) = \mathbb{E}^\dagger_\varepsilon[\Lambda_T^{\theta,\varepsilon}\, Z_T^\varepsilon \mid \mathcal{Y}_T] \ , \tag{1}$$

where

$$\Lambda_t^{\theta,\varepsilon} = \exp\{\frac{1}{\varepsilon^2} \int_0^t b^*(\theta, X_s)\, dX_s - \frac{1}{2\varepsilon^2} \int_0^t |b(\theta, X_s)|^2\, ds\} \ ,$$

for any $0 \leq t \leq T$, and where $\mathbb{P}^\dagger_\varepsilon$ is a probability measure on Ω, equivalent to $\mathbb{P}^\dagger_{\theta,\varepsilon}$ with Radon–Nikodym derivative

$$\frac{d\mathbb{P}^\dagger_{\theta,\varepsilon}}{d\mathbb{P}^\dagger_\varepsilon} = \Lambda^{\theta,\varepsilon}_T \ ,$$

so that under $\mathbb{P}^\dagger_\varepsilon$, $\{X_t, 0 \leq t \leq T\}$ and $\{Y_t, 0 \leq t \leq T\}$ are two independent Wiener processes, with covariance matrices $\varepsilon^2 \, I_m$ and $\varepsilon^2 \, I_d$ respectively.

By definition, the residual ζ_ε is the gradient (column–vector) of the log-likelihood function w.r.t. the parameter θ, evaluated at the nominal value $\theta = \alpha$, and suitably normalized, i.e.

$$\zeta_\varepsilon = (\, \varepsilon \, \partial \log L_\varepsilon(\alpha) \,)^* \ .$$

It follows from (1) that

$$\partial \, L_\varepsilon(\theta) = \mathbb{E}^\dagger_\varepsilon[\partial \log \Lambda^{\theta,\varepsilon}_T \, \Lambda^{\theta,\varepsilon}_T \, Z^\varepsilon_T \mid \mathcal{Y}_T] = \mathbb{E}^\dagger_{\theta,\varepsilon}[\partial \log \Lambda^{\theta,\varepsilon}_T \, Z^\varepsilon_T \mid \mathcal{Y}_T] \ ,$$

hence

$$\begin{aligned} \varepsilon \, \partial \, \log L_\varepsilon(\theta) = \frac{\varepsilon \, \partial \, L_\varepsilon(\theta)}{L_\varepsilon(\theta)} &= \frac{\varepsilon \, \mathbb{E}^\dagger_{\theta,\varepsilon}[\partial \, \log \Lambda^{\theta,\varepsilon}_T \, Z^\varepsilon_T \mid \mathcal{Y}_T]}{\mathbb{E}^\dagger_{\theta,\varepsilon}[Z^\varepsilon_T \mid \mathcal{Y}_T]} \\ &= \varepsilon \, \mathbb{E}_{\theta,\varepsilon}[\partial \, \log \Lambda^{\theta,\varepsilon}_T \mid \mathcal{Y}_T] = (\, \mathbb{E}_{\theta,\varepsilon}[\Xi^{\theta,\varepsilon}_T \mid \mathcal{Y}_T] \,)^* \ , \end{aligned}$$

where

$$\begin{aligned} \Xi^{\theta,\varepsilon}_t &= \int_0^t \partial \, b^*(\theta, X_s) \, \frac{1}{\varepsilon} \, [dX_s - b(\theta, X_s) \, ds] \\ &= \int_0^t \partial \, b^*(\theta, X_s) \, dW^\theta_s \ , \end{aligned}$$

for any $0 \leq t \leq T$.

Let $p^{\theta,\varepsilon}_t$ denote the unnormalized conditional density of the r.v. X_t given $\mathcal{Y}_t$ under $\mathbb{P}_{\theta,\varepsilon}$, and let $w^{\theta,\varepsilon}_t$ denote its derivative w.r.t. the parameter θ, i.e.

$$\int \phi(x) \, p^{\theta,\varepsilon}_t(x) \, dx = \mathbb{E}^\dagger_{\theta,\varepsilon}[\phi(X_t) \, Z^\varepsilon_t \mid \mathcal{Y}_t] \ ,$$

and

$$\int \phi(x) \, w^{\theta,\varepsilon}_t(x) \, dx = \mathbb{E}^\dagger_{\theta,\varepsilon}[\phi(X_t) \, \partial \log \Lambda^{\theta,\varepsilon}_t \, Z^\varepsilon_t \mid \mathcal{Y}_t] \ ,$$

for any test function ϕ defined on $\mathbb{R}^m$. Then the likelihood function can be expressed as

$$L_\varepsilon(\theta) = \mathbb{E}^\dagger_{\theta,\varepsilon}[Z^\varepsilon_T \mid \mathcal{Y}_T] = \int p^{\theta,\varepsilon}_T(x) \, dx \ , \tag{2}$$

and it follows from Campillo and LeGland [4, Section 4] that the residual can be expressed as

$$\zeta_\varepsilon = \mathbb{E}_{\alpha,\varepsilon}[\Xi_T^{\alpha,\varepsilon} \mid \mathcal{Y}_T] = \frac{[\displaystyle\int \varepsilon\, w_T^{\alpha,\varepsilon}(x)\, dx\,]^*}{\displaystyle\int p_T^{\alpha,\varepsilon}(x)\, dx} \; . \tag{3}$$

Moreover, $\{p_t^{\theta,\varepsilon}\,,\, 0 \leq t \leq T\}$ satisfies the Duncan–Mortensen–Zakai (DMZ) equation

$$\begin{cases} dp_t^{\theta,\varepsilon}(x) = \frac{1}{2}\,\varepsilon^2 \displaystyle\sum_{i,j=1}^m \frac{\partial^2}{\partial x_i \partial x_j}\, p_t^{\theta,\varepsilon}(x)\, dt - \sum_{i=1}^m \frac{\partial}{\partial x_i}\,[\, b^i(\theta,x)\, p_t^{\theta,\varepsilon}(x)\,]\, dt \\ \qquad\qquad\qquad + \dfrac{1}{\varepsilon^2} h^*(x)\, p_t^{\theta,\varepsilon}(x)\, dY_t \;, \\ p_0^{\theta,\varepsilon}(x) = \dfrac{1}{\varepsilon^m (2\pi)^{m/2}} \; \exp\{-\dfrac{|x|^2}{2\,\varepsilon^2}\} \;, \end{cases}$$

and $\{w_t^{\theta,\varepsilon}\,,\, 0 \leq t \leq T\}$ satisfies the stochastic partial differential equation

$$\begin{cases} dw_t^{\theta,\varepsilon}(x) = \frac{1}{2}\,\varepsilon^2 \displaystyle\sum_{i,j=1}^m \frac{\partial^2}{\partial x_i \partial x_j}\, w_t^{\theta,\varepsilon}(x)\, dt - \sum_{i=1}^m \frac{\partial}{\partial x_i}\,[\, b^i(\theta,x)\, w_t^{\theta,\varepsilon}(x)\,]\, dt \\ \qquad\qquad\qquad + \dfrac{1}{\varepsilon^2}\, h^*(x)\, w_t^{\theta,\varepsilon}(x)\, dY_t \\ \qquad\qquad\qquad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial x_i}\,[\, \partial b^i(\theta,x)\, p_t^{\theta,\varepsilon}(x)\,]\, dt \;, \\ w_0^{\theta,\varepsilon}(x) = 0 \;. \end{cases}$$

3 Small Noise Expansion

Let $(X^{(0)}, Y^{(0)})$ and $(X^{(1)}, Y^{(1)})$ denote the solutions of the following limiting ordinary (deterministic) differential system

$$\begin{cases} \dot{X}_t^{(0)} = b(\alpha, X_t^{(0)}) \;, \quad X_0^{(0)} = 0 \;, \\ \dot{Y}_t^{(0)} = h(X_t^{(0)}) \;, \quad Y_0^{(0)} = 0 \;, \end{cases}$$

and *linear tangent* stochastic differential system

$$\begin{cases} dX_t^{(1)} = b'(\alpha, X_t^{(0)})\, X_t^{(1)}\, dt + dW_t^\alpha \;, \quad X_0^{(1)} = \xi \;, \\ dY_t^{(1)} = h'(X_t^{(0)})\, X_t^{(1)}\, dt + dV_t^\alpha \;, \quad Y_0^{(1)} = 0 \;, \end{cases} \tag{4}$$

respectively. Define also

$$\bar{X}_t^\varepsilon = \frac{1}{\varepsilon}\,(X_t - X_t^{(0)})\ , \quad \text{and} \quad \bar{Y}_t^\varepsilon = \frac{1}{\varepsilon}\,(Y_t - Y_t^{(0)})\ ,$$

for any $0 \leq t \leq T$ and any $\varepsilon > 0$. Then $(\bar{X}^\varepsilon, \bar{Y}^\varepsilon)$ is the solution of the following *contaminated* stochastic differential system under $\mathbb{P}_{\theta,\varepsilon}$

$$\begin{cases} d\bar{X}_t^\varepsilon = \dfrac{1}{\varepsilon}\,[b(\theta, X_t^{(0)} + \varepsilon\,\bar{X}_t^\varepsilon) - b(\alpha, X_t^{(0)})]\,dt + dW_t^\theta\ , \quad \bar{X}_0^\varepsilon = \xi\ , \\ d\bar{Y}_t^\varepsilon = \dfrac{1}{\varepsilon}\,[h(X_t^{(0)} + \varepsilon\,\bar{X}_t^\varepsilon) - h(X_t^{(0)})]\,dt + dV_t^\theta\ , \quad \bar{Y}_0^\varepsilon = 0\ , \end{cases} \tag{5}$$

Notice that the mapping $u \longmapsto \dfrac{1}{\varepsilon}\,[b(\theta, X_t^{(0)} + \varepsilon\,u) - b(\alpha, X_t^{(0)})]$ is bounded and Lipschitz continuous, with a Lipschitz constant independent of $\varepsilon > 0$, hence the r.v. $\bar{X}_t^\varepsilon$ has moments of any order, which are uniform in $\varepsilon > 0$.

Under the nominal behaviour of the system, we have the following well–known result, see Freidlin and Wentzell [8].

Lemma 1. *As* $\varepsilon \downarrow 0$

$$\sup_{0 \leq t \leq T} |\bar{X}_t^\varepsilon - X_t^{(1)}| \longrightarrow 0\ , \quad \textit{and} \quad \sup_{0 \leq t \leq T} |\bar{Y}_t^\varepsilon - Y_t^{(1)}| \longrightarrow 0\ ,$$

in $\mathbb{P}_{\alpha,\varepsilon}$*–probability.*

Set

$$\phi_t = \frac{1}{\varepsilon^2} \int_0^t h^*(X_s^{(0)})\,dY_s - \frac{1}{\varepsilon^2} \int_0^t |h(X_s^{(0)})|^2\,ds\ ,$$

and define

$$\bar{p}_t^{\theta,\varepsilon}(u) = \varepsilon^m\ \exp\{-\phi_t\}\ p_t^{\theta,\varepsilon}(X_t^{(0)} + \varepsilon\,u)\ ,$$

$$\bar{w}_t^{\theta,\varepsilon}(u) = \varepsilon^{m+1}\ \exp\{-\phi_t\}\ w_t^{\theta,\varepsilon}(X_t^{(0)} + \varepsilon\,u)\ ,$$

Then by the Itô lemma, $\{\bar{p}_t^{\theta,\varepsilon}\ , 0 \leq t \leq T\}$ and $\{\bar{w}_t^{\theta,\varepsilon}\ , 0 \leq t \leq T\}$ satisfy the stochastic partial differential equations

$$\begin{cases} d\bar{p}_t^{\theta,\varepsilon}(u) = \frac{1}{2} \displaystyle\sum_{i,j=1}^m \frac{\partial^2}{\partial u_i \partial u_j}\,\bar{p}_t^{\theta,\varepsilon}(u)\,dt \\ \qquad\qquad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial u_i}(\,\frac{1}{\varepsilon}\,[b^i(\theta, X_t^{(0)} + \varepsilon\,u) - b^i(\alpha, X_t^{(0)})]\,\bar{p}_t^{\theta,\varepsilon}(u)\,)\,dt \\ \qquad\qquad + \dfrac{1}{\varepsilon^2}\,[h(X_t^{(0)} + \varepsilon\,u) - h(X_t^{(0)})]^*\,\bar{p}_t^{\theta,\varepsilon}(u)\ [dY_t - h(X_t^{(0)})\,dt]\ , \\ \bar{p}_0^{\theta,\varepsilon}(u) = \dfrac{1}{(2\pi)^{m/2}}\ \exp\{-\frac{1}{2}\,|u|^2\}\ , \end{cases}$$

and

$$\begin{cases} d\bar{w}_t^{\theta,\varepsilon}(u) = \tfrac{1}{2} \displaystyle\sum_{i,j=1}^m \frac{\partial^2}{\partial u_i \partial u_j} \bar{w}_t^{\theta,\varepsilon}(u)\, dt \\ \qquad\qquad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial u_i} (\frac{1}{\varepsilon} [b^i(\theta, X_t^{(0)} + \varepsilon\, u) - b^i(\alpha, X_t^{(0)})]\, \bar{w}_t^{\theta,\varepsilon}(u))\, dt \\ \qquad\qquad + \displaystyle\frac{1}{\varepsilon^2} [h(X_t^{(0)} + \varepsilon\, u) - h(X_t^{(0)})]^* \, \bar{w}_t^{\theta,\varepsilon}(u)\, [dY_t - h(X_t^{(0)})\, dt] \;, \\ \qquad\qquad - \displaystyle\sum_{i=1}^m \frac{\partial}{\partial u_i} [\partial b^i(\theta, X_t^{(0)} + \varepsilon\, u)\, \bar{p}_t^{\theta,\varepsilon}(u)\,]\, dt \;, \\ \bar{w}_0^{\theta,\varepsilon}(u) = 0 \;, \end{cases}$$

respectively. Let $\bar{\mathcal{Y}}_T^\varepsilon$ denote the σ–algebra generated by the process $\bar{Y}^\varepsilon$. It follows from Theorem 3.7 in Charalambous, Elliott and Krishnamurthy [6], that

$$\frac{\displaystyle\int \bar{p}_T^{\theta,\varepsilon}(u)\, du}{\displaystyle\int \bar{p}_T^{\alpha,\varepsilon}(u)\, du} = \mathbb{E}_{\alpha,\varepsilon}[\bar{\Upsilon}_T^{\theta,\varepsilon} \mid \bar{\mathcal{Y}}_T^\varepsilon] \;, \tag{6}$$

where

$$\bar{\Upsilon}_t^{\theta,\varepsilon} = \exp\{ \int_0^t \frac{1}{\varepsilon} [b(\theta, X_s^{(0)} + \varepsilon\, \bar{X}_s^\varepsilon) - b(\alpha, X_s^{(0)} + \varepsilon\, \bar{X}_s^\varepsilon)]^* \, dW_s^\alpha \\ - \tfrac{1}{2} \int_0^t |\frac{1}{\varepsilon} [b(\theta, X_s^{(0)} + \varepsilon\, \bar{X}_s^\varepsilon) - b(\alpha, X_s^{(0)} + \varepsilon\, \bar{X}_s^\varepsilon)|^2 \, ds\} \;,$$

for any $0 \leq t \leq T$. Similarly, it follows from Theorem 3.2 and Remark 3.3 in Charalambous, Elliott and Krishnamurthy [6], that

$$\frac{[\displaystyle\int \bar{w}_T^{\alpha,\varepsilon}(u)\, du\,]^*}{\displaystyle\int \bar{p}_T^{\alpha,\varepsilon}(u)\, du} = \mathbb{E}_{\alpha,\varepsilon}[\bar{\Xi}_T^{\alpha,\varepsilon} \mid \bar{\mathcal{Y}}_T^\varepsilon] \;, \tag{7}$$

where

$$\bar{\Xi}_t^{\alpha,\varepsilon} = \int_0^T \partial\, b^*(\alpha, X_s^{(0)} + \varepsilon\, \bar{X}_s^\varepsilon)\, dW_s^\alpha = \Xi_t^{\alpha,\varepsilon} \;,$$

for any $0 \leq t \leq T$.

4 Asymptotic Normality of the Residual Under the Nominal Hypothesis

Using (7) and the change of variable $x = X_T^{(0)} + \varepsilon u$ in (3), yields

$$\zeta_\varepsilon = \frac{[\int \varepsilon\, w_T^{\alpha,\varepsilon}(x)\, dx]^*}{\int p_T^{\alpha,\varepsilon}(x)\, dx} = \frac{[\int \varepsilon^{m+1}\, w_T^{\alpha,\varepsilon}(X_T^{(0)} + \varepsilon\, u)\, du]^*}{\int \varepsilon^m\, p_T^{\alpha,\varepsilon}(X_T^{(0)} + \varepsilon\, u)\, du}$$
$$= \frac{[\int \bar{w}_T^{\alpha,\varepsilon}(u)\, du]^*}{\int \bar{p}_T^{\alpha,\varepsilon}(u)\, du} = \mathbb{E}_{\alpha,\varepsilon}[\bar{\Xi}_T^{\alpha,\varepsilon} \mid \bar{\mathcal{Y}}_T^\varepsilon] \; .$$

To study the asymptotic behaviour of ζ_ε under $\mathbb{P}_{\alpha,\varepsilon}$ as $\varepsilon \downarrow 0$, we extend the contaminated model (5) and the linear tangent model (4) as follows

$$\begin{cases} d\bar{X}_t^\varepsilon = \dfrac{1}{\varepsilon}\, [\, b(\alpha, X_t^{(0)} + \varepsilon\, \bar{X}_t^\varepsilon) - b(\alpha, X_t^{(0)})\,]\, dt + dW_t^\alpha \; , \quad \bar{X}_0 = \xi \; , \\ d\bar{\Xi}_t^{\alpha,\varepsilon} = \partial b^*(\alpha, X_t^{(0)} + \varepsilon\, \bar{X}_t^\varepsilon)\, dW_t^\alpha \; , \quad \bar{\Xi}_0^{\alpha,\varepsilon} = 0 \; , \\ d\bar{Y}_t^\varepsilon = \dfrac{1}{\varepsilon}\, [\, h(X_t^{(0)} + \varepsilon\, \bar{X}_t^\varepsilon) - h(X_t^{(0)})\,]\, dt + dV_t^\alpha \; , \quad \bar{Y}_0^\varepsilon = 0 \; , \end{cases}$$

and

$$\begin{cases} dX_t^{(1)} = b'(\alpha, X_t^{(0)}) X_t^{(1)} dt + dW_t^\alpha \; , \quad X_0^{(1)} = \xi \; , \\ d\Xi_t^{(1)} = \partial b^*(\alpha, X_t^{(0)})\, dW_t^\alpha \; , \quad \Xi_0^{(1)} = 0 \; , \\ dY_t^{(1)} = h'(X_t^{(0)}) X_t^{(1)} dt + dV_t^\alpha \; , \quad Y_0^{(1)} = 0 \; , \end{cases} \tag{8}$$

respectively.

Lemma 2. *As $\varepsilon \downarrow 0$*

$$\sup_{0 \le t \le T} |\bar{\Xi}_t^{\alpha,\varepsilon} - \Xi_t^{(1)}| \longrightarrow 0 \; ,$$

in $\mathbb{P}_{\alpha,\varepsilon}$–probability.

PROOF. Notice that

$$|\partial b(\alpha, X_t^{(0)} + \varepsilon\, u) - \partial b(\alpha, X_t^{(0)})| \le K\, \varepsilon\, |u| \; ,$$

for any $u \in \mathbb{R}^m$, and the Doob inequality yields

$$\mathbb{E}_{\alpha,\varepsilon}[\sup_{0 \le t \le T} |\bar{\Xi}_t^{\alpha,\varepsilon} - \Xi_t^{(1)}|^2\,] \le 4\, K^2\, \varepsilon^2\, \mathbb{E}_{\alpha,\varepsilon}[\int_0^T |\bar{X}_t^\varepsilon|^2\, dt] \; ,$$

which goes to zero as $\varepsilon \downarrow 0$, since

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sup_{0\leq t\leq T} \mathbb{E}_{\alpha,\varepsilon}|\bar{X}_t^\varepsilon|^2 < \infty \ . \quad \square$$

Let $\mathcal{G}$ denote the σ–algebra generated by $(X^{(1)}, \Xi^{(1)}, \bar{Y}^\varepsilon)$, and let $\bar{\mathbb{Q}}_\varepsilon^{(1)}$ and $\mathbb{Q}^{(1)}$ denote the probability distribution of the r.v.'s $(X^{(1)}, \Xi^{(1)}, \bar{Y}^\varepsilon)$ and $(X^{(1)}, \Xi^{(1)}, Y^{(1)})$ respectively, on the canonical space $C([0,T]\,;\,\mathbb{R}^{m+p+d})$.

Lemma 3. *As* $\varepsilon \downarrow 0$

$$\|\bar{\mathbb{Q}}_\varepsilon^{(1)} - \mathbb{Q}^{(1)}\|_{\mathrm{TV}} \longrightarrow 0 \ .$$

Proof. It holds

$$\frac{d\mathbb{Q}^{(1)}}{d\bar{\mathbb{Q}}_\varepsilon^{(1)}} = \mathbb{E}_{\alpha,\varepsilon}[\exp\{\bar{M}_T^\varepsilon - \tfrac{1}{2}\,\langle \bar{M}^\varepsilon\rangle_T\} \mid \mathcal{G}] \ ,$$

where the martingale $\{\bar{M}_t^\varepsilon\,,\,0 \leq t \leq T\}$ is defined by

$$\begin{aligned}
\bar{M}_t^\varepsilon &= \int_0^t (\,h'(X_s^{(0)})\,X_s^{(1)} - \frac{1}{\varepsilon}\,[\,h(X_s^{(0)} + \varepsilon\,\bar{X}_s^\varepsilon) - h(X_s^{(0)})\,]\,)^*\,dV_s^\alpha \\
&= -\int_0^t (\,\frac{1}{\varepsilon}\,[\,h(X_s^{(0)} + \varepsilon\,\bar{X}_s^\varepsilon) - h(X_s^{(0)})\,] - h'(X_s^{(0)})\,\bar{X}_s^\varepsilon\,)^*\,dV_s^\alpha \\
&\qquad - \int_0^t (\,h'(X_s^{(0)})\,(\bar{X}_s^\varepsilon - X_s^{(1)})\,)^*\,dV_s^\alpha \ ,
\end{aligned}$$

for any $0 \leq t \leq T$. Notice that

$$|\frac{1}{\varepsilon}\,[\,h(X_t^{(0)} + \varepsilon\,u) - h(X_t^{(0)})\,] - h'(X_t^{(0)})\,u| \leq K\,\varepsilon\,|u|^2 \ ,$$

for any $u \in \mathbb{R}^m$, and

$$|h'(X_t^{(0)})\,(\bar{X}_t^\varepsilon - X_t^{(1)})\,| \leq K\,|\bar{X}_t^\varepsilon - X_t^{(1)}| \ ,$$

hence

$$\langle \bar{M}^\varepsilon\rangle_T \leq 2\,K^2\,\varepsilon^2 \int_0^T |\bar{X}_t^\varepsilon|^4\,dt + 2\,K^2\,T \sup_{0\leq t\leq T} |\bar{X}_t^\varepsilon - X_t^{(1)}|^2 \ ,$$

which goes to zero in $\mathbb{P}_{\alpha,\varepsilon}$–probability as $\varepsilon \downarrow 0$, since

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sup_{0\leq t\leq T} \mathbb{E}_{\alpha,\varepsilon}|\bar{X}_t^\varepsilon|^4 < \infty \ ,$$

and using Lemma 1. It follows from Problem 1.9.2 in Liptser and Shiryayev [12] that the same convergence holds for $\bar{M}_T^\varepsilon$, hence

$$\bar{Z}_T^\varepsilon = \exp\{\bar{M}_T^\varepsilon - \tfrac{1}{2}\,\langle \bar{M}^\varepsilon\rangle_T\} \longrightarrow 1 \ ,$$

in $\mathbb{P}_{\alpha,\varepsilon}$–probability as $\varepsilon \downarrow 0$. Therefore, using the Jensen inequality and the Scheffé theorem yields

$$\begin{aligned}\|\bar{\mathbb{Q}}^{(1)}_\varepsilon - \mathbb{Q}^{(1)}\|_{\mathrm{TV}} &= \mathbb{E}_{\alpha,\varepsilon}|1 - \frac{d\mathbb{Q}^{(1)}}{d\bar{\mathbb{Q}}^{(1)}_\varepsilon}| \\ &= \mathbb{E}_{\alpha,\varepsilon}|\mathbb{E}_{\alpha,\varepsilon}[1 - \bar{Z}^\varepsilon_T \mid \mathcal{G}]\,| \leq \mathbb{E}_{\alpha,\varepsilon}|1 - \bar{Z}^\varepsilon_T| \longrightarrow 0 \ . \quad \square\end{aligned}$$

The convergence in distribution of ζ_ε follows from Lemmas 2 and 3, and from Theorem 1.2 in Kleptsina, Liptser and Serebrovski [9].

Theorem 1. *As* $\varepsilon \downarrow 0$

$$\zeta_\varepsilon = \mathbb{E}_{\alpha,\varepsilon}[\bar{\varXi}^{\alpha,\varepsilon}_T \mid \bar{\mathcal{Y}}^\varepsilon_T] \Longrightarrow \mathbb{E}_\alpha[\varXi^{(1)}_T \mid \mathcal{Y}^{(1)}_T] = \zeta \ ,$$

under $\mathbb{P}_{\alpha,\varepsilon}$.

To characterize the limit r.v. ζ, the extended linear tangent model (8) is rewritten as

$$\left\{\begin{array}{l} d\begin{pmatrix} X^{(1)}_t \\ \varXi^{(1)}_t \end{pmatrix} = \begin{pmatrix} F_t & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^{(1)}_t \\ \varXi^{(1)}_t \end{pmatrix} dt + \begin{pmatrix} I \\ J^*_t \end{pmatrix} dW^\alpha_t, \quad \begin{pmatrix} X^{(1)}_0 \\ \varXi^{(1)}_0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \\ \qquad dY^{(1)}_t = \begin{pmatrix} H_t & 0 \end{pmatrix}\begin{pmatrix} X^{(1)}_t \\ \varXi_t \end{pmatrix} dt + dV^\alpha_t, \quad Y^{(1)}_0 = 0, \end{array}\right.$$

with the notations $F_t = b'(\alpha, X^{(0)}_t)$, $H_t = h'(X^{(0)}_t)$ and $J_t = \partial b(\alpha, X^{(0)}_t)$, and the solution of the above filtering problem yields

$$\zeta = \mathbb{E}_\alpha[\varXi^{(1)}_T \mid \mathcal{Y}^{(1)}_T] = \int_0^T r^*_t\, H^*_t\, [dY^{(1)}_t - H_t\, \hat{X}^{(1)}_t\, dt] \ , \tag{9}$$

and

$$\mathbb{E}_\alpha[(\varXi^{(1)}_T - \zeta)\,(\varXi^{(1)}_T - \zeta)^* \mid \mathcal{Y}^{(1)}_T] = \int_0^T (J^*_t\, J_t - r^*_t\, H^*_t\, H_t\, r_t)\, dt \ , \tag{10}$$

where $\{r_t\, , \, 0 \leq t \leq T\}$ satisfies the ODE

$$\dot{r}_t = F_t\, r_t - P_t\, H^*_t\, H_t\, r_t + J_t \ , \quad r_0 = 0 \ ,$$

and where $\hat{X}^{(1)}_t = \mathbb{E}_\alpha[X^{(1)}_t \mid \mathcal{Y}^{(1)}_t]$ and P_t are respectively the Kalman filter estimate and its asociated error covariance matrix in the linear tangent model (4), for any $0 \leq t \leq T$. Furthermore, the innovation process is a standard Wiener process, hence the r.v. ζ is Gaussian, with mean 0 and covariance matrix

$$\varSigma = \int_0^T r^*_t\, H^*_t\, H_t\, r_t\, dt \ .$$

Theorem 2. *The residual* ζ_ε *is asymptotically normal under the nominal hypothesis, i.e. as* $\varepsilon \downarrow 0$

$$\zeta_\varepsilon \Longrightarrow \mathcal{N}(0,\Sigma) \;, \quad \text{under } \mathbb{P}_{\alpha,\varepsilon} .$$

5 Local Asymptotic Normality (LAN)

To study the asymptotic normality of the residual under the contiguous alternative hypothesis, we first prove the local asymptotic normality property of the family of probability distributions of the observation process, a result which is of interest by itself.

As is well known [11], we can express the likelihood ratio as

$$\mathbb{E}_{\alpha,\varepsilon}[\frac{d\mathbb{P}_{\theta,\varepsilon}}{d\mathbb{P}_{\alpha,\varepsilon}} \mid \mathcal{Y}_T] = \frac{L_\varepsilon(\theta)}{L_\varepsilon(\alpha)} = \frac{\displaystyle\int p_T^{\theta,\varepsilon}(x)\,dx}{\displaystyle\int p_T^{\alpha,\varepsilon}(x)\,dx} \;. \tag{11}$$

Taking $\theta = \alpha + \varepsilon\Delta$, using (6) and the change of variable $x = X_T^{(0)} + \varepsilon\, u$ in the expression (11), yields

$$\begin{aligned} Z_\varepsilon(\Delta) = \frac{L_\varepsilon(\alpha+\varepsilon\,\Delta)}{L_\varepsilon(\alpha)} &= \frac{\displaystyle\int p_T^{\alpha+\varepsilon\,\Delta,\varepsilon}(x)\,dx}{\displaystyle\int p_T^{\alpha,\varepsilon}(x)\,dx} \\ &= \frac{\displaystyle\int \varepsilon^m\, p_T^{\alpha+\varepsilon\,\Delta,\varepsilon}(X_T^{(0)}+\varepsilon\,u)\,du}{\displaystyle\int \varepsilon^m\, p_T^{\alpha,\varepsilon}(X_T^{(0)}+\varepsilon u)\,du} \\ &= \frac{\displaystyle\int \bar{p}_T^{\alpha+\varepsilon\,\Delta,\varepsilon}(u)\,du}{\displaystyle\int \bar{p}_T^{\alpha,\varepsilon}(u)\,du} = \mathbb{E}_{\alpha,\varepsilon}[\bar{\Upsilon}_T^{\alpha+\varepsilon\,\Delta,\varepsilon} \mid \bar{\mathcal{Y}}_T^\varepsilon] \;. \end{aligned}$$

To study the joint asymptotic behaviour of $(\zeta_\varepsilon, Z_\varepsilon(\Delta))$ under $\mathbb{P}_{\alpha,\varepsilon}$ as $\varepsilon \downarrow 0$, we further extend the contaminated model (5) and the linear tangent model (4) as follows

$$\begin{cases} d\bar{X}_t^\varepsilon = \dfrac{1}{\varepsilon}\,[\,b(\alpha, X_t^{(0)}+\varepsilon\,\bar{X}_t^\varepsilon) - b(\alpha, X_t^{(0)})\,]\,dt + dW_t^\alpha \;, \quad \bar{X}_0^\varepsilon = \xi \;, \\ d\bar{\Xi}_t^{\alpha,\varepsilon} = \partial b^*(\alpha, X_t^{(0)}+\varepsilon\,\bar{X}_t^\varepsilon)\,dW_t^\alpha \;, \quad \bar{\Xi}_0^{\alpha,\varepsilon} = 0 \;, \\ d\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon} = \dfrac{1}{\varepsilon}\,[b(\alpha+\varepsilon\,\Delta, X_t^{(0)}+\varepsilon\,\bar{X}_t^\varepsilon) \\ \qquad\qquad - b(\alpha, X_t^{(0)}+\varepsilon\,\bar{X}_t^\varepsilon)]^*\,\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon}\,dW_t^\alpha \;, \quad \bar{\Upsilon}_0^{\alpha,\varepsilon} = 0 \;, \\ d\bar{Y}_t^\varepsilon = \dfrac{1}{\varepsilon}\,[\,h(X_t^{(0)}+\varepsilon\,\bar{X}_t^\varepsilon) - h(X_t^{(0)})\,]\,dt + dV_t^\alpha \;, \quad \bar{Y}_0^\varepsilon = 0 \;, \end{cases}$$

and

$$\begin{cases} dX_t^{(1)} = b'(\alpha, X_t^{(0)}) X_t^{(1)} dt + dW_t^\alpha \ , \quad X_0^{(1)} = \xi \ , \\ d\Xi_t^{(1)} = \partial b^*(\alpha, X_t^{(0)}) \, dW_t^\alpha \ , \quad \Xi_0^{(1)} = 0 \ , \\ d\Upsilon_t^{(1)} = \Delta^* \, \partial b^*(\alpha, X_t^{(0)}) \, \Upsilon_t^{(1)} \, dW_t^\alpha \ , \quad \Upsilon_0^{(1)} = 0 \ , \\ dY_t^{(1)} = h'(X_t^{(0)}) X_t^{(1)} dt + dV_t^\alpha \ , \quad Y_0^{(1)} = 0 \ , \end{cases}$$

respectively.

Lemma 4. *As* $\varepsilon \downarrow 0$

$$\sup_{0 \leq t \leq T} |\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon} - \Upsilon_t^{(1)}| \longrightarrow 0 \ ,$$

in $\mathbb{P}_{\alpha,\varepsilon}$*–probability.*

Proof. It holds

$$\begin{aligned} &d(\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon} - \Upsilon_t^{(1)}) \\ &\quad = (\frac{1}{\varepsilon} \, [b(\alpha + \varepsilon\,\Delta, X_t^{(0)} + \varepsilon\,\bar{X}_t^\varepsilon) - b(\alpha, X_t^{(0)} + \varepsilon\,\bar{X}_t^\varepsilon)] \\ &\qquad\qquad - \partial b(\alpha, X_t^{(0)}) \, \Delta \,)^* \, \Upsilon_t^{(1)} \, dW_t^\alpha \\ &\qquad + \frac{1}{\varepsilon} \, [b(\alpha + \varepsilon\,\Delta, X_t^{(0)} + \varepsilon\,\bar{X}_t^\varepsilon) - b(\alpha, X_t^{(0)} + \varepsilon\,\bar{X}_t^\varepsilon)]^* \\ &\qquad\qquad (\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon} - \Upsilon_t^{(1)}) \, dW_t^\alpha \ . \end{aligned}$$

Notice that

$$\begin{aligned} &|\frac{1}{\varepsilon} \, [b(\alpha + \varepsilon\,\Delta, X_t^{(0)} + \varepsilon\,u) - b(\alpha, X_t^{(0)} + \varepsilon\,u)] - \partial b(\alpha, X_t^{(0)}) \, \Delta| \\ &\quad \leq |\frac{1}{\varepsilon} \, [b(\alpha + \varepsilon\,\Delta, X_t^{(0)} + \varepsilon\,u) - b(\alpha, X_t^{(0)} + \varepsilon\,u)] - \partial b(\alpha, X_t^{(0)} + \varepsilon\,u) \, \Delta| \\ &\qquad + | \, [\partial b(\alpha, X_t^{(0)} + \varepsilon\,u) - \partial b(\alpha, X_t^{(0)})] \, \Delta| \\ &\quad \leq K \, \varepsilon \, |\Delta|^2 + K \, \varepsilon \, |u| \, |\Delta| \ , \end{aligned}$$

and

$$|\frac{1}{\varepsilon} \, [b(\alpha + \varepsilon\,\Delta, X_t^{(0)} + \varepsilon\,u) - b(\alpha, X_t^{(0)} + \varepsilon\,u)] \, | \leq K \, |\Delta| \ ,$$

for any $u \in \mathbb{R}^m$, and the Doob inequality yields

$$\mathbb{E}_{\alpha,\varepsilon}[\sup_{0 \leq t \leq T} |\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon} - \Upsilon_t^{(1)}|^2 \,]$$

$$\leq 12\, K^2\, \varepsilon^2\, |\Delta|^4\, \mathbb{E}_{\alpha,\varepsilon}[\int_0^T |\Upsilon_t^{(1)}|^2\, dt]$$
$$+ 12\, K^2\, \varepsilon^2\, |\Delta|^2\, \mathbb{E}_{\alpha,\varepsilon}[\int_0^T |\bar{X}_t^\varepsilon|^2\, |\Upsilon_t^{(1)}|^2\, dt]$$
$$+ 12\, K^2\, |\Delta|^2\, \mathbb{E}_{\alpha,\varepsilon}[\int_0^T |\bar{\Upsilon}_t^{\alpha+\varepsilon\,\Delta,\varepsilon} - \Upsilon_t^{(1)}|^2\, dt] \;,$$

which goes to zero as $\varepsilon \downarrow 0$, using the Gronwall lemma, since

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sup_{0\leq t\leq T} \mathbb{E}_{\alpha,\varepsilon} |\bar{X}_t^\varepsilon|^4 < \infty \;,$$

and

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sup_{0\leq t\leq T} \mathbb{E}_{\alpha,\varepsilon} |\Upsilon_t^{(1)}|^4 < \infty \;. \quad \square$$

The joint convergence in distribution of $(\zeta_\varepsilon, Z_\varepsilon(\Delta))$ follows from Lemmas 2 and 4, from another instance of Lemma 3, and from Theorem 1.2 in Kleptsina, Liptser and Serebrovski [9].

Theorem 3. *As* $\varepsilon \downarrow 0$

$$\begin{aligned}(\zeta_\varepsilon, Z_\varepsilon(\Delta)) \;&=\; (\mathbb{E}_{\alpha,\varepsilon}[\bar{\Xi}_T^{\alpha,\varepsilon} \mid \bar{\mathcal{Y}}_T^\varepsilon], \mathbb{E}_{\alpha,\varepsilon}[\bar{\Upsilon}_T^{\alpha+\varepsilon\,\Delta,\varepsilon} \mid \bar{\mathcal{Y}}_T^\varepsilon]) \\ &\Longrightarrow (\mathbb{E}_\alpha[\Xi_T^{(1)} \mid \mathcal{Y}_T^{(1)}], \mathbb{E}_\alpha[\Upsilon_T^{(1)} \mid \mathcal{Y}_T^{(1)}]) = (\zeta, Z(\Delta)) \;,\end{aligned}$$

under $\mathbb{P}_{\alpha,\varepsilon}$.

To characterize the limit r.v. $Z(\Delta)$, notice that

$$\Upsilon_T^{(1)} = \exp\{\Delta^* \int_0^T J_t^*\, dW_t^\alpha - \tfrac{1}{2}\, \Delta^* \int_0^T J_t^*\, J_t\, dt\; \Delta\} \;,$$

where $J_t = \partial b(\alpha, X_t^{(0)})$, hence

$$\begin{aligned} Z(\Delta) &= \mathbb{E}_\alpha[\Upsilon_T^{(1)} \mid \mathcal{Y}_T^{(1)}] \\ &= \exp\{-\tfrac{1}{2}\, \Delta^* \int_0^T J_t^*\, J_t\, dt\; \Delta\}\, \mathbb{E}_\alpha[\exp\{\Delta^*\, \Xi_T^{(1)}\} \mid \mathcal{Y}_T^{(1)}] \;,\end{aligned}$$

and it follows from (9) and (10) that

$$\mathbb{E}_\alpha[\exp\{\Delta^*\, \Xi_T^{(1)}\} \mid \mathcal{Y}_T^{(1)}] = \exp\{\Delta^*\, \zeta + \tfrac{1}{2}\, \Delta^* \int_0^T J_t^*\, J_t\, dt\, \Delta - \tfrac{1}{2}\, \Delta^*\, \Sigma\, \Delta\} \;.$$

Therefore

$$Z(\Delta) = \exp\{\Delta^*\, \zeta - \tfrac{1}{2}\, \Delta^*\, \Sigma\, \Delta\} \;.$$

Thus, we have proved the following local asymptotic normality property.

Theorem 4. *The family of probability distributions of the observation process is locally asymptotically normal (LAN) at the point α as $\varepsilon \downarrow 0$, i.e. for any $\Delta \in \mathbb{R}^p$*

$$Z_\varepsilon(\Delta) = \mathbb{E}_{\alpha,\varepsilon}[\frac{d\mathbb{P}_{\theta,\varepsilon}}{d\mathbb{P}_{\alpha,\varepsilon}} \mid \mathcal{Y}_T] \Longrightarrow \exp\{\Delta^* \zeta - \tfrac{1}{2} \Delta^* \Sigma \Delta\} = Z(\Delta) \; ,$$

where $\zeta \sim \mathcal{N}(0, \Sigma)$, under $\mathbb{P}_{\alpha,\varepsilon}$.

6 Application to FDI

Under the contiguous alternative hypothesis $\theta = \alpha + \varepsilon \Delta$, for any test function ϕ defined on $\mathbb{R}^p$, it holds

$$\begin{aligned}
\mathbb{E}_{\alpha+\varepsilon\Delta,\varepsilon}[\phi(\zeta_\varepsilon)] &= \mathbb{E}_{\alpha,\varepsilon}[\phi(\zeta_\varepsilon) \, \frac{d\mathbb{P}_{\alpha+\varepsilon\Delta,\varepsilon}}{d\mathbb{P}_{\alpha,\varepsilon}}] \\
&= \mathbb{E}_{\alpha,\varepsilon}[\phi(\zeta_\varepsilon) \, Z_\varepsilon(\Delta)] \\
&\longrightarrow \mathbb{E}_\alpha[\phi(\zeta) \, \exp\{\Delta^* \zeta - \tfrac{1}{2} \Delta^* \Sigma \Delta\}] \; ,
\end{aligned}$$

where the convergence follows from Theorems 3 and 4. Let F and G denote the Gaussian distributions on $\mathbb{R}^p$ with (not necessarily invertible) covariance matrix Σ, and mean vector 0 and $\Sigma \Delta$ respectively. Since G is absolutely continuous w.r.t. F, with Radon–Nikodym derivative

$$\frac{dG}{dF}(x) = \exp\{\Delta^* x - \tfrac{1}{2} \Delta^* \Sigma \Delta\} \; ,$$

it holds

$$\begin{aligned}
&\mathbb{E}_\alpha[\phi(\zeta) \, \exp\{\Delta^* \zeta - \tfrac{1}{2} \Delta^* \Sigma \Delta\}] \\
&\quad = \int \phi(x) \, \exp\{\Delta^* x - \tfrac{1}{2} \Delta^* \Sigma \Delta\} \, F(dx) = \int \phi(x) \, G(dx) \; ,
\end{aligned}$$

i.e. the following result has been proved.

Theorem 5. *The residual ζ_ε is asymptotically normal under the contiguous alternative hypothesis, i.e. as $\varepsilon \downarrow 0$*

$$\zeta_\varepsilon \Longrightarrow \mathcal{N}(\Sigma \Delta, \Sigma) \; , \quad \textit{under } \mathbb{P}_{\alpha+\varepsilon\Delta,\varepsilon}.$$

Combining the results of Theorems 2 and 5, yields

$$\zeta_\varepsilon \Longrightarrow \zeta \sim \begin{cases} \mathcal{N}(0, \Sigma) \; , & \text{under } \mathbb{P}_{\alpha,\varepsilon}, \\[2ex] \mathcal{N}(\Delta \Sigma, \Sigma) \; , & \text{under } \mathbb{P}_{\alpha+\varepsilon\,\Delta,\varepsilon}, \end{cases}$$

where

$$\Sigma = \int_0^T r_t^* \, H_t^* \, H_t \, r_t \, dt \ ,$$

depends only on the nominal value α of the parameter. To detect a change in the mean vector of the limiting Gaussian r.v., one can use a simple χ^2 test

$$\zeta^* \, \Sigma^{-1} \, \zeta \lesseqgtr c \ , \tag{12}$$

where $c > 0$ is the threshold, provided the covariance matrix Σ is invertible. Plugging the residual ζ_ε in, which depends only on the nominal value α of the parameter and on observations $\{Y_t \, , \, 0 \leq t \leq T\}$ produced by the original dynamical system, yields the following test

$$\zeta_\varepsilon^* \, \Sigma^{-1} \, \zeta_\varepsilon \lesseqgtr c \ ,$$

where $c > 0$ is the threshold. To select the threshold, error probabilities for this test can be approximated by those of the test (12).

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