A PARTICLE IMPLEMENTATION OF THE RECURSIVE MLE FOR PARTIALLY OBSERVED DIFFUSIONS

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Abstract In this paper, the problem of identifying a hidden Markov model (HMM) with general state space, e.g. a partially observed diffusion process, is considered. A particle implementation of the recursive maximum likelihood estimator for a parameter in the transition kernel of the Markov chain is presented. The key assumption is that the derivative of the transition kernel w.r.t. the parameter has a probabilistic interpretation, suitable for Monte Carlo simulation. Examples are given to show that this assumption is satisfied in quite general situations. As a result, the linear tangent filter, i.e. the derivative of the filter w.r.t. the parameter, is absolutely continuous w.r.t. the filter and the idea is to jointly approximate the (prediction) filter and its derivative with the empirical probability distribution and with a weighted empirical distribution associated with the same and unique particle system. Application to the identification of a stochastic volatility model is presented.

Keywords: hidden Markov model, stochastic volatility model, nonlinear filter, linear tangent filter, particle filter, recursive MLE.

1. HIDDEN MARKOV MODEL

The state sequence $\{X_k\,,\, k \geq 0\}$ is a Markov chain taking values in the space E, with transition kernel $Q(x,dx')$, i.e.

$$\mathbb{P}[X_{k+1} \in dx' \mid X_k = x] = Q(x,dx') \ .$$

The kernel $Q(x,dx')$ could depend on a parameter, that should be either estimated, or monitored (i.e. changes w.r.t. a nominal value should be detected), however the dependence w.r.t. the parameter is not written explicitly, so as to avoid intricated notations. The following assumption is made

It is *easy* to *simulate* a r.v. X with probability distribution $Q(x,dx')$, even though the analytical expression of the kernel $Q(x,dx')$ is not known, or is so complicated that it is *pratically impossible* to *compute* such integrals as

$$Q\,\phi(x) = \int_E Q(x,dx')\,\phi(x') \ ,$$

or

$$Q\,\mu(dx') = \int_E \mu(dx)\,Q(x,dx') \ .$$

This is the case for instance if the Markov chain $\{X_k\,,\, k \geq 0\}$ is obtained by sampling a diffusion process $\{X'_t\,,\, t \geq 0\}$ at discrete time instants $\{t_k\,,\, k \geq 0\}$, i.e. if $X_k = X'_{t_k}$, with

$$dX'_t = b(X'_t)\,dt + \sigma(X'_t)\,dW_t \ , \qquad (1)$$

where $\{W_t\,,\, t \geq 0\}$ is a Brownian motion. In this case, to *simulate* a r.v. with probability distribution $Q(x,dx')$ simply reduces to simulate (with an appropriate numerical discretisation scheme) the solution at time t_{k+1} of the stochastic differential equation (1) starting from the initial condition $X'_{t_k} = x$ at time t_k.

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The state sequence $\{X_k\,,\,k\geq 0\}$ is not observed, but instead an observation sequence $\{Y_k\,,\,k\geq 0\}$ is available, which has the following property : given the hidden states $\{X_k\,,\,k\geq 0\}$, the observations $\{Y_k\,,\,k\geq 0\}$ are mutually independent, and the conditional probability distribution of Y_k depends only on the hidden state X_k at the same time instant, and by definition

$$\mathbb{P}[Y_k \in dy \mid X_k = x] = g(x,y)\,\lambda(dy) \;,$$

and

$$\Psi_k(x) = g(x,Y_k) \;.$$

This *memoryless channel* assumption is satisfied for instance in the case where

$$Y_k = h(X_k) + V_k \;,$$

where $\{V_k\,,\,k\geq 0\}$ is a white noise sequence (i.e. a sequence of mutually independent r.v.'s) with probability distribution $q(v)\,dv$, independent of $\{X_k\,,\,k\geq 0\}$. In this case

$$\mathbb{P}[Y_k \in dy \mid X_k = x] = q(y-h(x))\,dy \;,$$

and

$$\Psi_k(x) = q(Y_k - h(x)) \;.$$

The *memoryless channel* assumption is also satisfied in the case where the covariance of the observation noise depends on the hidden state, i.e. in the case where

$$Y_k = r(X_k)\,V_k \;,$$

where $\{V_k\,,\,k\geq 0\}$ is a white noise sequence with probability distribution $q(v)\,dv$, independent of $\{X_k\,,\,k\geq 0\}$. In this case, provided the matrix $r(x)$ is invertible for any $x\in\mathbb{R}^m$, it holds

$$\mathbb{P}[Y_k \in dy \mid X_k = x] = \frac{q([r(x)]^{-1}\,y)}{\det r(x)}\,dy \;,$$

and

$$\Psi_k(x) = \frac{q([r(x)]^{-1}\,Y_k)}{\det r(x)} \;.$$

Given observations, the objective of nonlinear filtering is to estimate the hidden states, and to this effect the probability distributions

$$\mu_k(dx) = \mathbb{P}[X_k \in dx \mid Y_0,\cdots,Y_k] \;,$$

and

$$\mu_{k|k-1}(dx) = \mathbb{P}[X_k \in dx \mid Y_0,\cdots,Y_{k-1}] \;,$$

are introduced. The evolution of the sequence $\{\mu_k\,,\,k\geq 0\}$ taking values in the space of probability distributions on E, is very easily described by the following steps

$$\mu_{k-1} \longrightarrow \mu_{k|k-1} = Q\,\mu_{k-1}$$

$$\longrightarrow \mu_k = \Psi_k \cdot \mu_{k|k-1} \;,$$

where

$$\mu_{k|k-1}(dx') = Q\,\mu_{k-1}(dx') = \int_E \mu_{k-1}(dx)\,Q(x,dx') \;,$$

can happen to be *difficult* (if not just impossible) to *compute*, and where $\cdot$ denotes the projective product, i.e.

$$\mu_k(dx) = \Psi_k \cdot \mu_{k|k-1}(dx) = \frac{\Psi_k(x)\,\mu_{k|k-1}(dx)}{\langle \mu_{k|k-1}, \Psi_k\rangle} \;.$$

2. LINEAR TANGENT KERNEL / EXTENDED KERNEL, ETC.

If the transition kernel $Q(x,dx')$ depends on a parameter, then the filter μ_k depends also on the parameter, and one would like to compute the linear tangent filter w_k, i.e. the derivative of the filter μ_k w.r.t. the parameter. To this end, one needs first to study the linear tangent kernel $\Gamma(x,dx')$, i.e. the derivative of the transition kernel $Q(x,dx')$ w.r.t. the parameter, and the following assumption is made

Assumprion AC : The following probabilistic representation holds for the linear tangent kernel $\Gamma(x,dx')$

$$\Gamma\,\phi(x) = \int_E \Gamma(x,dx')\,\phi(x')$$

$$= \mathbb{E}[\phi(X_{k+1})\,\Xi_{k+1} \mid X_k = x] \;,$$

where $\{\,(X_k,\Xi_k)\,,\,k\geq 0\}$ is a Markov chain taking values in the product space $E\times F$, such that

$$\mathbb{P}[X_{k+1}\in dx', \Xi_{k+1}\in ds' \mid X_k = x, \Xi_k = s]$$

$$= K(x,dx',ds') \;.$$

The following assumption, which extends the similar assumption introduced in Section 1, is made

It is *easy* to *simulate* a r.v. (X,Ξ) with probability distribution $K(x,dx',ds')$, even though the analytical expression of the kernel $K(x,dx',ds')$ is not known, or is so complicated that it is *pratically impossible* to *compute* such integrals as

$$\Gamma\,\phi(x) = \int_{E\times F} s'\,\phi(x')\,K(x,dx',ds') \;,$$

or

$$\Gamma\,\mu(dx') = \int_{E\times F} \mu(dx)\,s'\,K(x,dx',ds') \;.$$

Example 2.1. Let the Markov chain $\{X_k\,,\,k\geq 0\}$ taking values in $E=\mathbb{R}^m$, be defined by

$$X_{k+1} = f(X_k) + g(X_k)\,W_k \;,$$

where only the function f depends on the parameter, and where $\{W_k\,,\,k\geq 0\}$ is a sequence of independent r.v.'s taking values in $\mathbb{R}^q$ with probability distribution $p(w)\,dw$. In the simple case where $g(x)=I$ for any $x\in\mathbb{R}^m$, the transition kernel $Q(x,dx')$ is given by

$$Q(x,dx') = p(x'-f(x))\,dx' \;,$$

and one can show directly that

$$\Gamma(x,dx') = \frac{-p'}{p}(x' - f(x))\, \partial f(x)\, Q(x,dx') \; ,$$

where ∂f denotes the derivative of the function f w.r.t. the parameter, i.e. Assumption AC is satisfied, with

$$\Xi_{k+1} = \frac{-p'}{p}(W_k)\, \partial f(X_k)$$

This result generalizes to the case where for any $x \in \mathbb{R}^m$ the matrix $g(x)$ has full rank, and the vector $\partial f(x)$ belongs to the range of $g(x)$.

Example 2.2. Let the Markov chain $\{X_k \,, k \geq 0\}$ taking values in $E = \mathbb{R}^m$, be defined by

$$X_{k+1} = f(X_k) + c\, g(X_k)\, W_k \; ,$$

where $\{W_k \,, k \geq 0\}$ is a sequence of independent r.v.'s taking values in $\mathbb{R}^q$ with probability distribution $p(w)\, dw$. In the simple case where $g(x) = I$ for any $x \in \mathbb{R}^m$, the transition kernel $Q(x,dx')$ is given by

$$Q(x,dx') = p(\frac{x' - f(x)}{c})\; \frac{dx'}{c^q} \; ,$$

and one can show directly that

$$\Gamma(x,dx') =$$

$$= [\, \frac{-p'}{p}(\frac{x' - f(x)}{c})\; \frac{x' - f(x)}{c^2} - \frac{q}{c}\,]\; Q(x,dx') \; ,$$

i.e. Assumption AC is satisfied, with

$$\Xi_{k+1} = \frac{1}{c}\, [\, \frac{-p'}{p}(W_k)\, W_k - q\,] \; ,$$

and

$$\Xi_{k+1} = \frac{1}{c}\, (\, |W_k|^2 - q\,) \; ,$$

in the special case where W_k is a zero mean Gaussian r.v. with identity covariance matrix. This results actually holds without any assumption on the matrix g.

Notice that in the above two examples, the r.v. Ξ_{k+1} depends only on (X_k, W_k), in which case it does not seem necessary to simulate Ξ_{k+1} in addition to W_k. In some cases, it is even possible to express W_k in terms of (X_k, X_{k+1}), and finally the r.v. $\Xi_{k+1} = I(X_k, X_{k+1})$ depends only on (X_k, X_{k+1}). This apparently very particular situation is actually very general, as the following result shows.

Lemma 2.3. Under Assumption AC

$$\Gamma(x,dx') = I(x,x')\; Q(x,dx') \; ,$$

with

$$I(x,x') = \mathbb{E}[\Xi_{k+1} \mid X_k = x, X_{k+1} = x'] \; ,$$

for any $x, x' \in E$.

However, and as the following example shows, there exist situations where the existence of the function I does not imply that an easy–to–compute explicit expression exists, whereas in opposition the joint simulation of (X_{k+1}, Ξ_{k+1}) is easy.

Example 2.4. Let the Markov chain $\{X_k \,, k \geq 0\}$ be defined by sampling at discrete time instants $\{t_k \,, k \geq 0\}$ a diffusion process $\{X'_t \,, t \geq 0\}$, i.e. $X_k = X'_{t_k}$, with

$$dX'_t = b(X'_t)\, dt + \sigma(X'_t)\, dW_t \; ,$$

where only the drift function b depends on the parameter, and where $\{W_t \,, t \geq 0\}$ is a Brownian motion. If for any $x \in \mathbb{R}^m$, the matrix $\sigma(x)$ has full rank, and the vector $\partial b(x)$ belongs to the range of $\sigma(x)$, then Assumption AC is satisfied, with

$$\Xi_{k+1} = \int_{t_k}^{t_{k+1}} [\, \partial b(X'_t)\,]^*$$

$$\sigma(X'_t)\; [\sigma^*(X'_t)\, \sigma(X'_t)]^{-1}\; dW_t \; ,$$

where ∂b denotes the derivative of the drift function b w.r.t. the parameter, see (Cérou *et al.*, 2001) or (Fournié *et al.*, 1999) for the simpler case where for any $x \in \mathbb{R}^m$ the matrix $\sigma(x)$ is invertible. It is easy (with an appropriate numerical discretization scheme) to jointly simulate (X_{k+1}, Ξ_{k+1}), but in opposition there does not exist in general a simple analytical expression for

$$I(x,x') = \mathbb{E}[\Xi_{k+1} \mid X_k = x, X_{k+1} = x'] \; .$$

By definition

$$\Gamma(x,dx') = \int_F s'\, K(x,dx',ds') \; ,$$

and

$$Q(x,dx') = \int_F K(x,dx',ds') \; .$$

On the product space $E \times E \times F$, define the projection $\pi_0 \;: (x,x',s') \longmapsto x$ on the (first) space E, the projection $\pi \;: (x,x',s') \longmapsto x'$ on the (second) space E and the projection $\pi_F \;: (x,x',s') \longmapsto s'$ on the auxiliary space F. For any probability distribution μ on the space E, the probability distribution $\mu \otimes K$ is defined on the product space $E \times E \times F$ by

$$(\mu \otimes K)(dx,dx',ds') = \mu(dx)\, K(x,dx',ds') \; .$$

It follows that

$$Q\mu = (\mu \otimes K) \circ \pi^{-1} \; ,$$

and

$$\Gamma\mu = (\pi_F\, (\mu \otimes K)) \circ \pi^{-1} \; ,$$

and

$$(w \ll \mu \quad \Longrightarrow \quad Q w = ((\frac{dw}{d\mu} \circ \pi_0)\; (\mu \otimes K)) \circ \pi^{-1}\,) \; .$$

Lemma 2.5. Under Assumption AC, $\Gamma\mu \ll Q\mu$ for any probability distribution μ on E, with Radon–Nikodym derivative (which depends on μ)

$$\frac{d(\Gamma\mu)}{d(Q\mu)}(x') = \mathbb{E}_\mu[\Xi_{n+1} \mid X_{n+1} = x'] \; .$$

Lemma 2.6. If the finite signed measure w is absolutely continuous w.r.t. the probability distribution μ, then $Q w \ll Q \mu$, with Radon–Nikodym derivative

$$\frac{d(Qw)}{d(Q\mu)}(x') = \mathbb{E}_\mu \left[\frac{dw}{d\mu}(X_k) \mid X_{k+1} = x' \right] .$$

By definition

$$F_k(\mu)\, w = \frac{\Psi_k\, w}{\langle \mu, \Psi_k \rangle} - \frac{\langle w, \Psi_k \rangle}{\langle \mu, \Psi_k \rangle}\; \frac{\Psi_k\, \mu}{\langle \mu, \Psi_k \rangle} ,$$

is the derivative at point μ and in the direction w, of the mapping $\mu \longmapsto \Psi_k \cdot \mu$.

Lemma 2.7. If the finite signed measure w is absolutely continuous w.r.t. the probability distribution μ, then $F_k(\mu)\, w \ll \Psi_k \cdot \mu$, with Radon–Nikodym derivative

$$\frac{d(F_k(\mu)\, w)}{d(\Psi_k \cdot \mu)}(x) = \frac{dw}{d\mu}(x) - \langle \Psi_k \cdot \mu, \frac{dw}{d\mu} \rangle .$$

3. PARTICLE APPROXIMATION OF SOME FINITE SIGNED MEASURES

With the notations of the previous section, it easily seen that the probability distribution $Q\mu$ and the finite signed measures $\Gamma \mu$ and $Q w$ can be put in the general form $(r\,(\mu \otimes K)) \circ \pi^{-1}$ for some appropriate choice of the weight function r, namely $r \equiv 1$, $r = \pi_F$ and $r = \dfrac{dw}{d\mu} \circ \pi_0$ respectively. The weighted particle approximation of a finite signed measure of the general form $r\,(\mu \otimes K)$ is defined by

$$r\,(\mu \otimes K) \approx r\, S^N(\mu \otimes K)$$

$$= \frac{1}{N} \sum_{i=1}^N r(\xi_0^i, \xi^i, \Xi^i)\; \delta_{(\xi_0^i, \xi^i, \Xi^i)} ,$$

where the r.v.'s $\{ \xi_0^i, \xi^i, \Xi^i ,\, i = 1, \cdots, N \}$ form an N–sample with probability distribution $\mu \otimes K$, which can be easily achieved in the following manner : independently for any $i = 1, \cdots, N$

$$\xi_0^i \sim \mu(dx) ,$$

and

$$(\xi^i, \Xi^i) \sim K(\xi_0^i, dx', ds') ,$$

and the corresponding particle approximation for the marginal measure $(r\,(K \otimes \mu)) \circ \pi^{-1}$ is defined by

$$(r\,(\mu \otimes K)) \circ \pi^{-1} \approx (r\, S^N(\mu \otimes K)) \circ \pi^{-1}$$

$$= \frac{1}{N} \sum_{i=1}^N r(\xi_0^i, \xi^i, \Xi^i)\; \delta_{\xi^i} .$$

In particular for the weight functions $r \equiv 1$, $r = \pi_F$ and $r = \dfrac{dw}{d\mu} \circ \pi_0$, it holds

$$Q\mu = (\mu \otimes K) \circ \pi^{-1} \approx \frac{1}{N} \sum_{i=1}^N \delta_{\xi^i} ,$$

$$\Gamma \mu = (\pi_F\,(\mu \otimes K)) \circ \pi^{-1} \approx \frac{1}{N} \sum_{i=1}^N \Xi^i\, \delta_{\xi^i} ,$$

and

$$Q w = ((\frac{dw}{d\mu} \circ \pi_0)\,(\mu \otimes K)) \circ \pi^{-1} \approx \frac{1}{N} \sum_{i=1}^N \frac{dw}{d\mu}(\xi_0^i)\; \delta_{\xi^i} ,$$

respectively. For any test function ϕ defined on E, it holds

$$\sup_{\|\phi\|=1} \mathbb{E} | \frac{1}{N} \sum_{i=1}^N \phi(\xi^i) - \langle Q\mu, \phi \rangle | \leq \frac{1}{\sqrt{N}} ,$$

$$\sup_{\|\phi\|=1} \mathbb{E} | \frac{1}{N} \sum_{i=1}^N \Xi^i\, \phi(\xi^i) - \langle \Gamma\mu, \phi \rangle |$$
$$\leq \frac{1}{\sqrt{N}} \, \{ \sup_{x \in E} \int_{E \times F} |s'|^2\, K(x, dx', ds') \}^{1/2} ,$$

and

$$\sup_{\|\phi\|=1} \mathbb{E} | \frac{1}{N} \sum_{i=1}^N \frac{dw}{d\mu}(\xi_0^i)\; \phi(\xi^i) - \langle Q w, \phi \rangle |$$
$$\leq \frac{1}{\sqrt{N}} \, \{ \int_E |\frac{dw}{d\mu}(x)|^2\, \mu(dx) \}^{1/2} .$$

4. JOINT PARTICLE APPROXIMATION OF THE FILTER AND THE LINEAR TANGENT FILTER

Recall that the evolution of the sequence $\{ \mu_k ,\, k \geq 0 \}$ taking values in the space of probability distributions on E, is described by the following two steps

$$\mu_{k-1} \longrightarrow \mu_{k|k-1} = Q\, \mu_{k-1}$$
$$\longrightarrow \mu_k = \Psi_k \cdot \mu_{k|k-1} .$$

If w_k denotes at each time instant the linear tangent filter, i.e. the derivative of the filter μ_k w.r.t. the parameter, then the evolution of the sequence $\{ w_k ,\, k \geq 0 \}$ taking values in the linear tangent space to the space of probability distributions on E, i.e. taking values in the space of finite signed measures on E with zero total mass, is described by the following two steps, which are linear tangent versions of the prediction step and correction step respectively

$$w_{k-1} \longrightarrow w_{k|k-1} = Q\, w_{k-1} + \Gamma\, \mu_{k-1}$$
$$\longrightarrow w_k = F_k(\mu_{k|k-1})\, w_{k|k-1} .$$

Under Assumption AC, it is easily seen by induction, and using Lemmas 2.5, 2.6 and 2.7, that at each time instant $w_{k|k-1} \ll \mu_{k|k-1}$ and $w_k \ll \mu_k$. In view of this absolute continuity property, and of the key assumption that it is *easy* to *simulate* r.v.'s with probability distribution $K(x,dx',ds')$, the idea is to jointly approximate the predictor $\mu_{k|k-1}$ and its derivative $w_{k|k-1}$ w.r.t. the parameter with the empirical probability distribution and a weighted empirical distribution associated with the same and unique N–sample, i.e.

$$\mu_{k|k-1} \approx \mu^N_{k|k-1} = \frac{1}{N} \sum_{i=1}^N \delta_{\xi^i_{k|k-1}} ,$$

and

$$w_{k|k-1} \approx w^N_{k|k-1} = \frac{1}{N} \sum_{i=1}^N \rho^i_{k|k-1} \, \delta_{\xi^i_{k|k-1}} .$$

With this definition $w^N_{k|k-1} \ll \mu^N_{k|k-1}$, with Radon–Nikodym derivative

$$r^N_{k|k-1}(x) = \frac{1}{|I^N_{k|k-1}(x)|} \sum_{i \in I^N_{k|k-1}(x)} \rho^i_{k|k-1} ,$$

where $I^N_{k|k-1}(x) = \{i = 1, \cdots, N : \xi^i_{k|k-1} = x\}$, for any x in the support supp $\mu^N_{k|k-1}$ of the discrete probability distribution $\mu^N_{k|k-1}$. Notice that in most cases, the particle locations $\{\xi^i_{k|k-1}, i = 1, \cdots, N\}$ happen to be all distinct, and the much simpler relation

$$r^N_{k|k-1}(\xi^i_{k|k-1}) = \rho^i_{k|k-1} ,$$

holds for any $i = 1, \cdots, N$. This approximation is completely characterized by the set $\{\xi^i_{k|k-1}, \rho^i_{k|k-1}, i = 1, \cdots, N\}$ of particles and weights, and the algorithm is completely described by the mechanism which builds $\{\xi^i_{k+1|k}, \rho^i_{k+1|k}, i = 1, \cdots, N\}$ from $\{\xi^i_{k|k-1}, \rho^i_{k|k-1}, i = 1, \cdots, N\}$. This mechanism is as follows :

(i) the correction step is applied *exactly* to $\mu^N_{k|k-1}$, which results in

$$\mu^N_k = \Psi_k \cdot \mu^N_{k|k-1} = \sum_{i=1}^N \frac{\Psi_k(\xi^i_{k|k-1}) \, \delta_{\xi^i_{k|k-1}}}{\displaystyle\sum_{j=1}^N \Psi_k(\xi^j_{k|k-1})} = \sum_{i=1}^N \omega^i_k \, \delta_{\xi^i_{k|k-1}} ,$$

and the linear tangent correction step is applied *exactly* to $w^N_{k|k-1}$, which results in

$$w^N_k = F_k(\mu^N_{k|k-1}) \, w^N_{k|k-1} = [\, r^N_{k|k-1} - \langle \mu^N_k, r^N_{k|k-1} \rangle \,] \, \mu^N_k ,$$

(ii) instead of trying to *compute* $Q \, \mu^N_k$, the following particle approximation

$$\mu^N_{k+1|k} = S^N(Q \, \mu^N_k) = \frac{1}{N} \sum_{i=1}^N \delta_{\xi^i_{k+1|k}} ,$$

is used, instead of trying to *compute*

$$Q \, w^N_k = Q(r^N_{k|k-1} \, \mu^N_k) - \langle \mu^N_k, r^N_{k|k-1} \rangle \, Q \, \mu^N_k = ((r^N_{k|k-1} \circ \pi_0) \, (\mu^N_k \otimes Q)) \circ \pi^{-1} - \langle \mu^N_k, r^N_{k|k-1} \rangle \, Q \, \mu^N_k ,$$

the following weighted particle approximation

$$(\, (r^N_{k|k-1} \circ \pi_0) \, S^N(\mu^N_k \otimes Q) \,) \circ \pi^{-1} - \langle S^N(\mu^N_k), r^N_{k|k-1} \rangle \, S^N(Q \, \mu^N_k) = \frac{1}{N} \sum_{i=1}^N [\, r^N_{k|k-1}(\xi^i_k) - \frac{1}{N} \sum_{j=1}^N r^N_{k|k-1}(\xi^j_k) \,] \, \delta_{\xi^i_{k+1|k}} ,$$

is used, and instead of trying to *compute*

$$\Gamma \, \mu^N_k = (\pi_F \, (\mu^N_k \otimes K)) \circ \pi^{-1} ,$$

the following weighted particle approximation

$$(\pi_F \, S^N(\mu^N_k \otimes K)) \circ \pi^{-1} = \frac{1}{N} \sum_{i=1}^N \Xi^i_{k+1} \, \delta_{\xi^i_{k+1|k}} ,$$

is used, hence finally the weighted particle approximation

$$w^N_{k+1|k} = \frac{1}{N} \sum_{i=1}^N [\, r^N_{k|k-1}(\xi^i_k) + \Xi^i_{k+1} - \frac{1}{N} \sum_{j=1}^N r^N_{k|k-1}(\xi^j_k) \,] \, \delta_{\xi^i_{k+1|k}} = \frac{1}{N} \sum_{i=1}^N \rho^i_{k+1|k} \, \delta_{\xi^i_{k+1|k}} ,$$

where the r.v.'s $\{\xi^i_k, \xi^i_{k+1|k}, \Xi^i_{k+1}, i = 1, \cdots, N\}$ form an N–sample with probability distribution $\mu^N_k \otimes K$, which can be easily achieved in the following manner : independently for any $i = 1, \cdots, N$

$$\xi^i_k \sim \mu^N_k(dx) ,$$

which is easy, since the probability distribution μ^N_k is discrete, and

$$(\xi^i_{k+1|k}, \Xi^i_{k+1}) \sim K(\xi^i_k, dx', ds') ,$$

which is easy, by assumption.

An alternate redistribution scheme has been proposed recently by (Doucet and Tadić, n.d.)

5. PARTICLE FILTER IMPLEMENTATION OF THE RECURSIVE MLE

In this section, the parameter is denoted by θ and dependence w.r.t. the parameter appears explicitly in the notation for the transition kernel $Q^\theta(x,dx')$, and for the linear tangent kernel $K^\theta(x,dx',ds')$. Monitoring the parametric model, i.e. detecting a small change from a nominal value, corresponding to the normal behaviour of the system, has been addressed in (Cérou and Le Gland, 2000). Another question is to identify the parametric model, and it is natural to consider the recursive MLE, which is defined by the following relation

$$\widehat{\theta}_k = \widehat{\theta}_{k-1} + \gamma_k \, \frac{\langle \widehat{w}_{k|k-1}, \Psi_k \rangle}{\langle \widehat{\mu}_{k|k-1}, \Psi_k \rangle} \; , \tag{2}$$

where typically $\gamma_k \simeq k^{-2/3}$, and the averaged estimator (which achieves the minimum variance of the estimation error) is obtained by post–processing

$$\overline{\theta}_k = \overline{\theta}_{k-1} + \frac{1}{k} \, (\widehat{\theta}_k - \overline{\theta}_{k-1}) \; .$$

Here, the adaptive filter $\{\widehat{\mu}_k \, , k \geq 0\}$ and the adaptive linear tangent filter $\{\widehat{w}_k \, , k \geq 0\}$ satisfy the same equations as the filter and the linear tangent filter respectively, in which the value of the parameter is adapted at each time instant according to equation (2), i.e.

$$\widehat{\mu}_{k-1} \longrightarrow \widehat{\mu}_{k|k-1} = Q^{\widehat{\theta}_{k-1}} \, \widehat{\mu}_{k-1}$$

$$\longrightarrow \widehat{\mu}_k = \Psi_k \cdot \widehat{\mu}_{k|k-1} \; ,$$

and

$$\widehat{w}_{k-1} \longrightarrow \widehat{w}_{k|k-1} = Q^{\widehat{\theta}_{k-1}} \, \widehat{w}_{k-1} + \Gamma^{\widehat{\theta}_{k-1}} \, \widehat{\mu}_{k-1}$$

$$\longrightarrow \widehat{w}_k = F_k(\widehat{\mu}_{k|k-1}) \, \widehat{w}_{k|k-1} \; ,$$

respectively. The particle implementation of the recursive MLE is

$$\widehat{\theta}^N_k = \widehat{\theta}^N_{k-1} + \gamma_k \, [\sum_{i=1}^N \rho^i_{k|k-1} \, \omega^i_k] \; ,$$

and

$$\overline{\theta}^N_k = \overline{\theta}^N_{k-1} + \frac{1}{k} \, (\widehat{\theta}^N_k - \overline{\theta}^N_{k-1}) \; .$$

The mathematical analysis of the asymptotic properties of the estimator $\widehat{\theta}^N_k$ as $k \to \infty$ and $N \to \infty$ is far beyond the scope of this paper, and would rely on joint stability properties of the filter and the linear tangent filter, which is a very difficult question. Even the asymptotic properties of the estimator $\widehat{\theta}_k$ as $k \to \infty$ are difficult to prove, unless some mixing assumption holds for the transition kernels $Q^\theta(x,dx')$ and the linear tangent kernels $\Gamma^\theta(x,dx')$, which practically implies that the state–space E should be compact, see e.g. (Douc and Matias, 2001) where only the non–recursive MLE is studied.

6. APPLICATION TO A STOCHASTIC VOLATILITY MODEL

The following stochastic volatility model, with mean–reverting hidden diffusion

$$dX'_t = a \, (b - X'_t) \, dt + c \, X'_t \, dW'_t \; , \qquad X'_0 > 0$$

$$dY'_t = \sqrt{X'_t} \, dV'_t \; ,$$

where $\{W'_t \, , t \geq 0\}$ and $\{V'_t \, , t \geq 0\}$ are independent Brownian motions, has been considered by (Genon-Catalot *et al.*, 2000). An alternate discrete–time observation model is considered in the present paper, in which an approximate Markov chain $\{X_k \, , k \geq 0\}$ is used, based on a Euler (or an alternate splitting–up) scheme, i.e.

$$X_k = (1 - a \, \Delta_k) \, X_{k-1} + a \, b \, \Delta_k + c \, X_{k-1} \, W_k \; ,$$

where $\Delta_k = t_{k+1} - t_k$ and where $\{W_k \, , k \geq 0\}$ is a Gaussian white noise sequence with variance Δ_k, and

$$Y_k = \sqrt{X_k} \, V_k \; ,$$

where $\{V_k \, , k \geq 0\}$ is a Gaussian white noise sequence independent of $\{W_k \, , k \geq 0\}$, hence

$$\Psi_k(x) = \frac{1}{\sqrt{x}} \, \exp\{-\frac{Y_k^2}{2x}\} \; , \qquad x > 0 \; .$$

In this case, it follows from Examples 2.1 and 2.2 that Assumption AC is satisfied, with

$$\Xi_{k+1} = (\frac{b - X_k}{c \, X_k} \, W_k \, , \, \frac{a}{c \, X_k} \, W_k \, , \, \frac{1}{c} \, (|W_k|^2 - 1)) \; .$$

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