USING NOISY GEOREFERENCED INFORMATION SOURCES FOR NAVIGATION AND TRACKING

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ABSTRACT

Localization, navigation and tracking form a special application domain of Bayesian filtering, where the position and velocity of a mobile (and possibly additional hyper–parameters) should be estimated based on (i) a prior model for the possible displacement of the mobile, (ii) noisy measurements provided by a sensor, and (iii) a georeferenced information source (digital map, reference data base, etc.), providing for each spatial position an estimate of the quantity measured by the sensor. For example in terrain–aided navigation (TAN) a radio–altimeter combined with an inertial navigation system (INS) provides an estimation of the terrain height below the platform, which can be correlated with the terrain height at each horizontal position, as read on a digital map. In wireless communications, the signal power received by the mobile from an access point (WiFi) or from a base station (GSM, UMTS) and measured by the mobile itself, can be correlated with another estimation of the signal power received at each spatial position, as read on a digital attenuation map or from a reference data base.

Values read on a digital map are usually subject to errors which are in general spatially correlated and modeled as Gaussian random fields, with a known correlation function. This results in a temporal correlation of measurement noises, which should be accounted for in evaluating the likelihood function, an essential step in the derivation of the equation for the Bayesian filter. The method described below shows how to perform this evaluation in an optimal way, using classical properties of Gaussian random vectors, and how to implement numerically the resulting Bayesian filter in terms of a particle filter.

1. SPATIAL ERROR MODEL

The value $\phi_{\rm map}(r)$ read on the digital map at the spatial position r differs from the true value $\phi(r)$ by a global bias b which is assumed independent of the position r and by an additive Gaussian noise $\varepsilon(r)$ with zero mean and variance $\sigma^2(r)$, and with correlation coefficient $\rho(r,r')$ depending on positions r and r', i.e.

$$\mathbb{E}[\varepsilon(r)\,\varepsilon(r')] = \sigma(r)\,\sigma(r')\,\rho(r,r') \;,$$

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with $-1 \leq \rho(r,r') \leq 1$. Notice that $\rho(r,r') = 1$ automatically reflects that errors due to the map at two different instants are the same if the mobile has the same spatial position at these two different instants. At time t_k, a value ϕ_k is provided by the sensor, with an additive non necessarily Gaussian error V_k, independent of the error due to the digital map. In view of

$$\phi_{\rm map}(r_k) = \phi(r_k) + b + \varepsilon(r_k) \;,$$

and

$$\phi_k = \phi(r_k) + V_k \;,$$

the following observation equation

$$\phi_k = \phi_{\rm map}(r_k) - b + V_k - \varepsilon(r_k) \;,$$

is obtained after elimination and relates the measurement ϕ_k with the unknown spatial position r_k.

Notice that in integrated navigation problems, the state variable is the inertial estimation error $\delta r_k = r_k - r_k^{\rm ins}$ rather than the raw position r_k, where the preliminary estimation $r_k^{\rm ins}$ in position is obtained as an output of INS, and accordingly the observation equation is usually written in the form

$$\phi_k = \phi_{\rm map}(r_k^{\rm ins} + \delta r_k) - b + V_k - \varepsilon(r_k^{\rm ins} + \delta r_k) \;,$$

with a prior model for the inertial estimation error δr_k.

The following model is considered, where hidden states $\{X_k\}$ form a Markov chain with values in a space E — not necessarily a Euclidean space — with initial probability distribution

$$\mathbb{P}[X_0 \in dx] = \eta_0(dx) \;,$$

and transition probability kernels

$$\mathbb{P}[X_k \in dx' \mid X_{k-1} = x] = Q_k(x,dx') \;.$$

Scalar observations $\{Y_k\}$ are related to hidden states by

$$Y_k = h_k(X_k) + V_k + \varepsilon(X_k) \;,$$

where $\{V_k\}$ is a sequence of independent — not necessarily Gaussian — random variables, and $\{\varepsilon(x)\,,\; x \in E\}$ is a scalar Gaussian random field with zero mean, variance $\sigma^2(x)$ and correlation function $\rho(x,x')$ for any $x, x' \in E$. It is assumed that the Markov chain $\{X_k\}$, the noise sequence $\{V_k\}$ and

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the Gaussian random field $\{\varepsilon(x)\,,\, x \in E\}$ are independent, but clearly the random variables $\{\varepsilon(X_k)\}$ are correlated, and are dependent on the Markov chain $\{X_k\}$, which makes the problem non standard. As a result, conditionnally w.r.t. the hidden sequence $X_{0:n} = (X_0, \cdots, X_n)$, the random vector $(\varepsilon(X_0), \cdots, \varepsilon(X_n))$ is Gaussian, with zero mean and covariance matrix

$$R_{0:n} = (R_{k,l})_{k,l=0,1,\cdots,n}\ ,$$

defined by

$$R_{k,l} = \sigma(X_k)\,\sigma(X_l)\,\rho(X_k, X_l)\ .$$

2. GAUSSIAN SENSOR NOISE

Consider first the case where the sensor noise $\{V_k\}$ is modeled as a Gaussian white noise sequence, and where V_k has zero mean and variance σ_k^2. Conditionnally on $X_{0:n} = x_{0:n}$, the observations $Y_{0:n} = (Y_0, \cdots, Y_n)$ form a Gaussian random vector, with mean vector

$$m_{0:n}(x_{0:n}) = (m_k(x_k))_{k=0,1,\cdots,n}\ ,$$

defined by

$$m_k(x_k) = h_k(x_k)\ ,$$

and covariance matrix

$$R^a_{0:n}(x_{0:n}) = (R^a_{k,l}(x_k, x_l))_{k,l=0,1,\cdots,n}\ ,$$

defined by

$$R^a_{k,l}(x_k, x_l) = \sigma_k^2\,\delta_{k,l} + \sigma(x_k)\,\sigma(x_l)\,\rho(x_k, x_l)\ .$$

2.1. Memoryless channel property

Introducing the block decomposition

$$R^a_{0:n}(x_{0:n}) = \begin{pmatrix} R^a_{0:n-1}(x_{0:n-1}) & S_n(x_{0:n}) \\ S^*_n(x_{0:n}) & \sigma^2(x_n) + \sigma_n^2 \end{pmatrix}\ ,$$

for the covariance matrix, it follows that a simple recursive expression holds for the inverse information matrix

$$\begin{aligned} I^a_{0:n}(x_{0:n}) &= R^{a\,-1}_{0:n}(x_{0:n}) \\ &= \begin{pmatrix} I & -T^a_n(x_{0:n}) \\ 0 & 1 \end{pmatrix} \\ &\qquad \begin{pmatrix} I^a_{0:n-1}(x_{0:n-1}) & 0 \\ 0 & 1/\Delta^a_n(x_{0:n}) \end{pmatrix} \\ &\qquad \begin{pmatrix} I & 0 \\ -T^{a\,*}_n(x_{0:n}) & 1 \end{pmatrix}\ , \end{aligned}$$

where by definition

$$T^a_n(x_{0:n}) = I^a_{0:n-1}(x_{0:n-1})\,S_n(x_{0:n})\ ,$$

and

$$\begin{aligned} \Delta^a_n(x_{0:n}) &= \sigma^2(x_n) + \sigma_n^2 \\ &\qquad - S^*_n(x_{0:n})\,I^a_{0:n-1}(x_{0:n-1})\,S_n(x_{0:n})\ , \end{aligned}$$

is the Schur complement of the matrix $R^a_{0:n-1}(x_{0:n-1})$ in the block–matrix $R^a_{0:n}(x_{0:n})$, hence

$$\begin{aligned} v^*_{0:n}\,I^a_{0:n}(x_{0:n})\,v_{0:n} &= \frac{|v_n - T^{a\,*}_n(x_{0:n})\,v_{0:n-1}|^2}{\Delta^a_n(x_{0:n})} \\ &\qquad + v^*_{0:n-1}\,I^a_{0:n-1}(x_{0:n-1})\,v_{0:n-1}\ , \end{aligned}$$

and in addition

$$\det I^a_{0:n}(x_{0:n}) = \frac{\det I^a_{0:n-1}(x_{0:n-1})}{\Delta^a_n(x_{0:n})}\ .$$

Therefore

$$\begin{aligned} &\mathbb{P}[Y_{0:n} \in dy_{0:n} \mid X_{0:n} = x_{0:n}] \\ &\qquad = \prod_{k=0}^{n} g_k(y_k, y_{0:k-1}, x_{0:k})\,dy_0 \cdots dy_n\ , \end{aligned}$$

with

$$\begin{aligned} g_k(y_k, y_{0:k-1}, x_{0:k}) &= \frac{1}{\sqrt{\Delta^a_k(x_{0:k})}} \\ \exp\{-\frac{1}{2\,\Delta^a_k(x_{0:k})}\,|\,y_k &- m_k(x_k) \\ - T^{a\,*}_k(x_{0:k})\,(y_{0:k-1} &- m_{0:k-1}(x_{0:k-1}))\,|^2\}\ . \end{aligned}$$

In other words, the classical memoryless channel property holds in path–space and the equation for the Bayesian filter results immediately, as well as its numerical implementation in terms of particle filters.

2.2. Bayesian filter

The idea is to introduce the Markov chain $\{X_{0:n}\}$ with values in path–space, with initial probability distribution

$$\mathbb{P}[X_{0:0} \in dx] = \eta_0(dx)\ ,$$

and transition probability kernels

$$\begin{aligned} &\mathbb{P}[X_{0:k} \in dx'_{0:k} \mid X_{0:k-1} = x_{0:k-1}] \\ &\qquad = \delta_{x_0}(dx'_0) \cdots \delta_{x_{k-1}}(dx'_{k-1})\,Q_k(x_{k-1}, dx'_k)\ , \end{aligned}$$

and to express the conditional probability distribution of the hidden state $X_{0:n}$ given observations $Y_{0:n}$ as

$$\langle \mu_{0:n}, \phi_{0:n} \rangle = \mathbb{E}[\phi_{0:n}(X_{0:n}) \mid Y_{0:n}] = \frac{\langle \gamma_{0:n}, \phi_{0:n} \rangle}{\langle \gamma_{0:n}, 1 \rangle}$$

where

$$\langle \gamma_{0:n}, \phi_{0:n} \rangle = \mathbb{E}[\phi_{0:n}(X_{0:n}) \prod_{k=0}^{n} g_k(X_{0:k})] ,$$

for any bounded measurable function $\phi_{0:n}$ defined on the path–space $E_{0:n} = E \times \cdots \times E$, and where

$$g_k(x_{0:k}) = g_k(Y_k, Y_{0:k-1}, x_{0:k}) ,$$

with the usual abuse of notation. The expectation in the above definition only concerns the random variables $X_{0:n}$, and the observations $Y_{0:n}$ are considered as parameters.

2.3. Particle approximation

The interpretation of the Bayesian filter in terms of Feynman–Kac flows in path–space [1] results in the following particle approximation

$$\mu_{0:n} \approx \sum_{i=1}^{N} w_n^i \, \delta_{\xi_{0:n}^i} \quad \text{with} \quad \sum_{i=1}^{N} w_n^i = 1 ,$$

in terms of the weighted empirical probability distribution associated with the particle system $(\xi_{0:n}^1, \cdots, \xi_{0:n}^N)$ in path–space. The resulting SIR algorithm can be described as follows :

For $k = 0$, independently for $i = 1, \cdots, N$

- simulate $\xi_{0:0}^i \sim \eta_0(dx)$,
- define the weight $w_0^i \propto g_0(\xi_{0:0}^i)$.

For any $k \geq 1$, independently for $i = 1, \cdots, N$

- simulate τ_k^i with values in $\{1, \cdots, N\}$ according to weights $(w_{k-1}^1, \cdots, w_{k-1}^N)$, and select $\widehat{\xi}_{0:k-1}^i$ accordingly as $\widehat{\xi}_{0:k-1}^i = \xi_{0:k-1}^{\tau_k^i}$,
- simulate $\xi_k^i \sim Q_k(\widehat{\xi}_{k-1}^i, dx')$, and define a new path $\xi_{0:k}^i$ as $\xi_{0:k}^i = (\widehat{\xi}_{0:k-1}^i, \xi_k^i)$,
- define the new weight $w_k^i \propto g_k(\xi_{0:k}^i)$.

3. NON GAUSSIAN SENSOR NOISE

In the general case, the sensor noise $\{V_k\}$ is modeled as a sequence of independent random variables, and V_k has probability distribution $q_k^V(v)\,dv$. Conditionnally on $X_{0:n} = x_{0:n}$ and $V_{0:n} = v_{0:n}$, the observations $Y_{0:n} = (Y_0, \cdots, Y_n)$ form a Gaussian random vector, with mean vector

$$m_{0:n}^a(x_{0:n}, v_{0:n}) = (m_k^a(x_k, v_k))_{k=0,1,\cdots,n}$$

defined by

$$m_k^a(x_k, v_k) = h_k(x_k) + v_k ,$$

and covariance matrix

$$R_{0:n}(x_{0:n}) = (R_{k,l}(x_k, x_l))_{k,l=0,1,\cdots,n} ,$$

defined by

$$R_{k,l}(x_k, x_l) = \sigma(x_k)\,\sigma(x_l)\,\rho(x_k, x_l) .$$

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$$R_{0:n}(x_{0:n}) = \begin{pmatrix} R_{0:n-1}(x_{0:n-1}) & S_n(x_{0:n}) \\ S_n^*(x_{0:n}) & \sigma^2(x_n) \end{pmatrix} ,$$

for the covariance matrix, it follows that a simple recursive expression holds for the inverse information matrix

$$\begin{aligned} I_{0:n}(x_{0:n}) &= R_{0:n}^{-1}(x_{0:n-1}) \\ &= \begin{pmatrix} I & -T_n(x_{0:n}) \\ 0 & 1 \end{pmatrix} \\ &\qquad \begin{pmatrix} I_{0:n-1}(x_{0:n-1}) & 0 \\ 0 & 1/\Delta_n(x_n) \end{pmatrix} \\ &\qquad \begin{pmatrix} I & 0 \\ -T_n^*(x_{0:n}) & 1 \end{pmatrix} , \end{aligned}$$

where by definition

$$T_n(x_{0:n}) = I_{0:n-1}(x_{0:n-1})\,S_n(x_{0:n}) ,$$

and

$$\Delta_n(x_{0:n}) = \sigma^2(x_n) - S_n^*(x_{0:n})\,I_{0:n-1}(x_{0:n-1})\,S_n(x_{0:n}) ,$$

is the Schur complement of the matrix $R_{0:n-1}(x_{0:n-1})$ in the block–matrix $R_{0:n}(x_{0:n})$, hence

$$\begin{aligned} v_{0:n}^*\,I_{0:n}(x_{0:n})\,v_{0:n} &= \frac{|v_n - T_n^*(x_{0:n})\,v_{0:n-1}|^2}{\Delta_n(x_{0:n})} \\ &\qquad + v_{0:n-1}^*\,I_{0:n-1}(x_{0:n-1})\,v_{0:n-1} , \end{aligned}$$

and in addition

$$\det I_{0:n}(x_{0:n}) = \frac{\det I_{0:n-1}(x_{0:n-1})}{\Delta_n(x_{0:n})} .$$

Therefore

$$\begin{aligned} &\mathbb{P}[Y_{0:n} \in dy_{0:n} \mid X_{0:n} = x_{0:n}, V_{0:n} = v_{0:n}] \\ &\quad = \prod_{k=0}^{n} g_k^a(y_k, y_{0:k-1}, x_{0:k}, v_{0:k})\,dy_0 \cdots dy_n , \end{aligned}$$

with

$$g_k^a(y_k, y_{0:k-1}, x_{0:k}, v_{0:k}) = \frac{1}{\sqrt{\Delta_k(x_{0:k})}}$$
$$\exp\{-\frac{1}{2\,\Delta_k(x_{0:k})}\,|\,y_k - m_k^a(x_k, v_k)$$
$$- T_k^*(x_{0:k})\,(y_{0:k-1} - m_{0:k-1}^a(x_{0:k-1}, v_{0:k-1}))\,|^2\}\,.$$

In other words, the classical memoryless channel property holds in path–space and the equation for the Bayesian filter results immediately, as well as its numerical implementation in terms of particle filters.

3.2. Bayesian filter

The idea is to introduce the Markov chain $\{(X_{0:n}, V_{0:n})\}$ with values in augmented path–space, with initial probability distribution

$$\mathbb{P}[X_{0:0} \in dx, V_{0,0} \in dv] = \eta_0(dx)\; q_0^V(v)\,dv\;,$$

and transition probability kernels

$$\begin{aligned}\mathbb{P}[X_{0:k} \in dx'_{0:k}, V_{0:k} \in dv'_{0:k} \mid X_{0:k-1} = x_{0:k-1},\\ V_{0:k-1} = v_{0:k-1}]\\ = \delta_{x_0}(dx'_0)\cdots\delta_{x_{k-1}}(dx'_{k-1})\; Q_k(x_{k-1}, dx'_k)\\ q_0^V(v'_0)\cdots q_k^V(v'_k)\; dv'_0\cdots dv'_k\;,\end{aligned}$$

and to express the conditional probability distribution of the hidden state $(X_{0:n}, V_{0:n})$ given observations $Y_{0:n}$ as

$$\langle\mu_{0:n}^a, \phi_{0:n}\rangle = \mathbb{E}[\phi_{0:n}(X_{0:n}, V_{0:n}) \mid Y_{0:n}] = \frac{\langle\gamma_{0:n}^a, \phi_{0:n}\rangle}{\langle\gamma_{0:n}^a, 1\rangle}$$

where

$$\langle\gamma_{0:n}^a, \phi_{0:n}\rangle = \mathbb{E}[\phi_{0:n}(X_{0:n}, V_{0:n})\;\prod_{k=0}^n g_k^a(X_{0:k}, V_{0:k})\,]\;,$$

for any bounded measurable function $\phi_{0:n}$ defined on the augmented path–space $E_{0:n}^a = E\times\cdots\times E\times\mathbb{R}^{n+1}$, and where

$$g_k^a(x_{0:k}, v_{0:k}) = g_k^a(Y_k, Y_{0:k-1}, x_{0:k}, v_{0:k})\;,$$

with the usual abuse of notation. The expectation in the above definition only concerns the random variables $X_{0:n}$ and $V_{0:n}$, and the observations $Y_{0:n}$ are considered as parameters.

3.3. Particle approximation

The interpretation of the Bayesian filter in terms of Feynman–Kac flows in augmented path–space [1] results in the following particle approximation

$$\mu_{0:n}^a \approx \sum_{i=1}^N w_n^i\;\delta_{(\xi_{0:n}^i,\;v_{0:n}^i)} \quad\text{with}\quad \sum_{i=1}^N w_n^i = 1\;,$$

in terms of the weighted empirical probability distribution associated with the particle system $(\xi_{0:n}^1, \cdots, \xi_{0:n}^N)$ in augmented path–space. The resulting SIR algorithm can be described as follows :

For $k = 0$, independently for $i = 1, \cdots, N$

- simulate $\xi_{0:0}^i \sim \eta_0(dx)$ and $v_{0:0}^i \sim q_0^V(v)\,dv$,
- define the weight $w_0^i \propto g_0^a(\xi_{0:0}^i, v_{0:0}^i)$.

For any $k \geq 1$, independently for $i = 1, \cdots, N$

- simulate τ_k^i with values in $\{1, \cdots, N\}$ according to weights $(w_{k-1}^1, \cdots, w_{k-1}^N)$, and select $(\widehat{\xi}_{0:k-1}^i, \widehat{v}_{0:k-1}^i)$ accordingly as $\widehat{\xi}_{0:k-1}^i = \xi_{0:k-1}^{\tau_k^i}$ and $\widehat{v}_{0:k-1}^i = v_{0:k-1}^{\tau_k^i}$,
- simulate $\xi_k^i \sim Q_k(\widehat{\xi}_{k-1}^i, dx')$ and $v_k^i \sim q_k^V(v)\,dv$, and define a new path $(\xi_{0:k}^i, v_{0:k}^i)$ as $\xi_{0:k}^i = (\widehat{\xi}_{0:k-1}^i, \xi_k^i)$ and $v_{0:k}^i = (\widehat{v}_{0:k-1}^i, v_k^i)$,
- define the new weight $w_k^i \propto g_k^a(\xi_{0:k}^i, v_{0:k}^i)$.

4. CONCLUDING REMARKS

The size of the problem, e.g. the dimension of the information matrices $I_{0:n}^a(x_{0:n})$ and $I_{0:n}(x_{0:n})$, increases with the time horizon n. This cannot be avoided, and it is the price to be paid to account for the possibility in a navigation or tracking problem that after a long excursion the mobile returns now to a region where it was before, which implies that the error due to the map now is the same as, or is highly correlated with, the error due to the map before.

Notice that in the special case of a Gaussian sensor noise, two different particle approximations have been presented, which depend on

- whether the sensor noise is incorporated in the mean vector $m_{0:n}^a(x_{0:n}, v_{0:n})$,
- or its variance is incorporated in the covariance matrix $R_{0:n}^a(x_{0:n})$.

5. REFERENCES

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