THRESHOLD SELECTION IN THE GLR TEST FOR CHANGE DETECTION IN PARTIALLY OBSERVED DIFFUSION PROCESSES

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1 Introduction

This paper is concerned with change detection in a partially observed diffusion model, i.e. detection of a change, occuring at some unknown change time, in the drift coefficient or the observation function. The problem is to decide, based on observations $\{Y_t\,,\,0\leq t\leq T\}$ in a finite time interval, between :

• The *null* hypothesis (H_0)

$$dX_t \;=\; b(X_t)\,dt+\varepsilon\,dW_t^\varepsilon\;,$$

$$dY_t \;=\; h(X_t)\,dt+\varepsilon\,dV_t^\varepsilon\;,$$

where $\{W_t^\varepsilon\,,\,t\geq 0\}$ and $\{V_t^\varepsilon\,,\,t\geq 0\}$ are independent Wiener processes under P_ε, and $X_0=\bar{x}$.

• The *alternate* (multiple) hypothesis (H_1) :
For some $0\leq\tau\leq T'$, with $T'<T$

$$dX_t \;=\; [b(X_t)+a(X_t)\,1_{(t\,\geq\,\tau)}]\,dt+\varepsilon\,dW_t^{\varepsilon,\tau}\;,$$

$$dY_t \;=\; [h(X_t)+g(X_t)\,1_{(t\,\geq\,\tau)}]\,dt+\varepsilon\,dV_t^{\varepsilon,\tau}\;,$$

where $\{W_t^{\varepsilon,\tau}\,,\,t\geq 0\}$ and $\{V_t^{\varepsilon,\tau}\,,\,t\geq 0\}$ are independent Wiener processes under $P_{\varepsilon,\tau}$, and $X_0=\bar{x}$.

The proposed approach is to use a generalized likelihood ratio (GLR) test, i.e. to consider the following region for rejecting the null hypothesis (H_0)

$$D_\varepsilon=\{\sup_{0\leq\tau\leq T'}\rho_\varepsilon(\tau)>c\}\;,$$

where c is a given threshold, and $\rho_\varepsilon(\tau)=\ell_\varepsilon(\tau)-\ell_\varepsilon(T)$ denotes a suitably normalized log–likelihood function for the estimation of the change time τ, based on observations $\{Y_t\,,\,0\leq t\leq T\}$.

The purpose of this paper is to prove that the threshold c can be chosen in a such way that both the probability of *false alarm* and the probability of *no detection* go to zero as $\epsilon\downarrow 0$. These probabilities are defined respectively as

$$F_\varepsilon=P_\varepsilon(D_\varepsilon)\quad\text{and}\quad M_\varepsilon=\sup_{0\leq\tau_0\leq T'}P_{\varepsilon,\tau_0}(D_\varepsilon^c)\;.$$

This result is an extension to the partially observed case, of the results obtained in Campillo, Kutoyants, LeGland [2].

2 Log–likelihood Function

It was proved in Campillo, LeGland [1], that the log–likelihood function for the estimation of the change time τ, based on observations $\{Y_t\,,\,0\leq t\leq T\}$, can be expressed as

$$\ell_\varepsilon(\tau)=\varepsilon^2\,\log\int_\Omega\exp\{\,\frac{1}{\varepsilon^2}\,F(\tau,\omega)\,\}\,dP_{\varepsilon,\tau}(\omega)$$

where $\Omega=C([0,T]\,;\mathbf{R}^m)$, and

$$F(\tau,\omega)=\int_0^T h^\tau(t,\omega_t)\,dY_t-\tfrac{1}{2}\int_0^T|h^\tau(t,\omega_t)|^2\,dt\;.$$

Throughout the paper, the following time–dependent coefficients, parametrized by the unknown parameter τ, are used

$$b^\tau(t,x)\;=\;b(x)+a(x)\,1_{(t\,\geq\,\tau)}\;,$$

$$h^\tau(t,x)\;=\;h(x)+g(x)\,1_{(t\,\geq\,\tau)}\;.$$

An efficient way of computing the GLR test statistics has been proposed in [1], using a pair of forward–backward SPDE's, one involving the coefficients before the change, the other involving the coefficients after the change. With this formulation, integration over the path space Ω is replaced by integration over the state space $\mathbf{R}^m$.

Indeed, consider the forward SPDE

$$d\mu_t^\varepsilon \; = \; \{ \; \tfrac{1}{2} \, \varepsilon^2 \; \Delta\mu_t^\varepsilon - \nabla(b \; \mu_t^\varepsilon) \; \} \, dt + \varepsilon^{-2} \; h \; \mu_t^\varepsilon \, dY_t \; ,$$

with Dirac initial condition $\mu_0^\varepsilon = \delta_{\bar{x}}$, and the backward SPDE

$$dv_t^\varepsilon + \{ \; \tfrac{1}{2} \, \varepsilon^2 \; \Delta v_t^\varepsilon + [b+a] \; \nabla v_t^\varepsilon \; \} \, dt$$

$$+ \varepsilon^{-2} \; [h+g] \; v_t^\varepsilon \; dY_t \; = \; 0 \; ,$$

with initial condition (at final time) $v_T^\varepsilon \equiv 1$.

Proposition 2.1 *The log–likelihood function $\rho_\varepsilon(\tau)$ can be computed as*

$$\rho_\varepsilon(\tau) = \varepsilon^2 \; \log \frac{\displaystyle\int_{\mathbf{R}^m} v_\tau^\varepsilon(x) \, \mu_\tau^\varepsilon(dx)}{\displaystyle\int_{\mathbf{R}^m} \mu_T^\varepsilon(dx)} \; ,$$

and the GLR test statistics as

$$\sup_{0 \le \tau \le T'} \rho_\varepsilon(\tau) = \varepsilon^2 \; \log \sup_{0 \le \tau \le T'} \frac{\displaystyle\int_{\mathbf{R}^m} v_\tau^\varepsilon(x) \, \mu_\tau^\varepsilon(dx)}{\displaystyle\int_{\mathbf{R}^m} \mu_T^\varepsilon(dx)} \; .$$

3 Large Deviations

The key idea to study the asymptotic behaviour of the error probabilities F_ε and M_ε when $\varepsilon \downarrow 0$, is to study the limiting expression of the GLR test statistics $\rho_\varepsilon(\tau)$ as $\varepsilon \downarrow 0$, using large deviations arguments similar to those in James, LeGland [3]. The situation can be summarized as follows.

- Under the null hypothesis (H_0)

(i) when $\varepsilon \downarrow 0$

$$P_\varepsilon(\sup_{0 \le t \le T} |Y_t - y_t| > \delta) \longrightarrow 0 \; ,$$

where $\{y_t \, , \, 0 \le t \le T\}$ is the output of the following limiting deterministic system

$$\begin{aligned} \dot{x}_t \; &= \; b(x_t) \; , \quad x_0 = \bar{x} \\ y_t \; &= \; h(x_t) \; . \end{aligned} \tag{1}$$

(ii) $\{P_\varepsilon \, , \, \varepsilon > 0\}$ satisfies a large deviations principle, with rate function

$$I(\phi) = \tfrac{1}{2} \int_0^T |\dot{\phi}_t - b(\phi_t)|^2 \, dt \; ,$$

if $\phi \in \Omega$ is absolutely continuous, $I(\phi) = +\infty$ otherwise.

- Under the alternate hypothesis (H_1) :

For all $0 \le \tau \le T'$, with $T' < T$

(i) when $\varepsilon \downarrow 0$

$$P_{\varepsilon,\tau}(\sup_{0 \le t \le T} |Y_t - y_t^\tau| > \delta) \longrightarrow 0 \; ,$$

where $\{y_t^\tau \, , \, 0 \le t \le T\}$ is the output of the following limiting deterministic system

$$\begin{aligned} \dot{x}_t^\tau \; &= \; b(x_t^\tau) + a(x_t^\tau) \, 1_{(t \ge \tau)} \; , \quad x_0^\tau = \bar{x} \\ y_t^\tau \; &= \; h(x_t^\tau) + g(x_t^\tau) \, 1_{(t \ge \tau)} \; . \end{aligned} \tag{2}$$

(ii) $\{P_{\varepsilon,\tau} \, , \, \varepsilon > 0\}$ satisfies a large deviations principle, with rate function

$$I(\tau,\phi) = \tfrac{1}{2} \int_0^T |\dot{\phi}_t - b^\tau(t,\phi_t)|^2 \, dt \; ,$$

if $\phi \in \Omega$ is absolutely continuous, $I(\tau,\phi) = +\infty$ otherwise.

These large deviations results are used in the next two sections, to obtain the limiting behaviour, under either the null hypothesis (H_0) or the alternate hypothesis (H_1), of the log–likelihood function $\rho_\varepsilon(\tau)$ as $\varepsilon \downarrow 0$. In both cases, the convergence result is stated in the form of a Varadhan theorem in probability, uniform w.r.t. the parameter τ (see Proposition 4.1 and Proposition 5.1 below).

4 Probability of False Alarm

If no change has occurred, the following limiting expression holds for the statistics $\rho_\varepsilon(\tau)$ under P_ε as $\varepsilon \downarrow 0$

$$\begin{aligned} \rho_0(\tau) \; = \; &- \inf_{\phi \, : \, \phi_0 = \bar{x}} [\; \tfrac{1}{2} \int_0^T |\dot{\phi}_t - b^\tau(t,\phi_t)|^2 \, dt \\ &\qquad + \tfrac{1}{2} \int_0^T |y_t - h^\tau(t,\phi_t)|^2 \, dt \;] \\ &+ \inf_{\phi \, : \, \phi_0 = \bar{x}} [\; \tfrac{1}{2} \int_0^T |\dot{\phi}_t - b(\phi_t)|^2 \, dt \end{aligned}$$

$$+ \tfrac{1}{2} \int_0^T |y_t - h(\phi_t)|^2 \, dt \ ]$$

$$= - \inf_{\phi \,:\, \phi_0 = \bar{x}} [\ \tfrac{1}{2} \int_0^T |\dot{\phi}_t - b^\tau(t, \phi_t)|^2 \, dt$$

$$+ \tfrac{1}{2} \int_0^T |y_t - h^\tau(t, \phi_t)|^2 \, dt \ ] \leq 0 \ ,$$

where $\{y_t \, , \, t \geq 0\}$ is the output of the limiting deterministic system (1). Notice that the following expression is also available for $\rho_0(\tau)$

$$\rho_0(\tau) \ = \ - \inf_{\phi \,:\, \phi_0 = \bar{x}} [\ \tfrac{1}{2} \int_0^\tau |\dot{\phi}_t - b(\phi_t)|^2 \, dt$$

$$+ \tfrac{1}{2} \int_0^\tau |y_t - h(\phi_t)|^2 \, dt$$

$$+ \tfrac{1}{2} \int_\tau^T |\dot{\phi}_t - [b + a](\phi_t)|^2 \, dt$$

$$+ \tfrac{1}{2} \int_\tau^T |y_t - [h + g](\phi_t)|^2 \, dt \ ] \ .$$

Proposition 4.1 *The following large deviations result holds*

$$P_\varepsilon(\sup_{0 \leq \tau \leq T'} |\rho_\varepsilon(\tau) - \rho_0(\tau)| > \delta) \longrightarrow 0 \ ,$$

when $\varepsilon \downarrow 0$.

Define

$$\rho_0^* = \sup_{0 \leq \tau \leq T'} \rho_0(\tau) \leq 0 \ .$$

Proposition 4.2 *The probability of false alarm* F_ε *goes to zero as* $\varepsilon \downarrow 0$, *provided the threshold* c *satisfies*

$$c > c_{\min} = \rho_0^* \ . \qquad (3)$$

The lower bound $c_{\min}$ can be computed exactly, using a pair of forward–backward dynamic programming equations, one involving the coefficients before the change, the other involving the coefficients after the change. With this formulation, optimization over the path space Ω is repaced by optimization over the state space $\mathbf{R}^m$.

Indeed, consider the forward dynamic programming equation

$$\frac{\partial S_t}{\partial t} = -\tfrac{1}{2} \, |\nabla S_t|^2 - b \, \nabla S_t + \tfrac{1}{2} \, |y_t - h|^2 \ ,$$

with initial condition $S_0(\bar{x}) = 0$, and $S_0(x) = \bar{S}$ (with $\bar{S}$ large enough) for $x \neq \bar{x}$, and the backward dynamic programming equation

$$\frac{\partial V_t}{\partial t} - \tfrac{1}{2} \, |\nabla V_t|^2 + [b + a] \, \nabla V_t + \tfrac{1}{2} \, |y_t - [h + g]|^2 = 0 \ ,$$

with initial condition (at final time) $V_T \equiv 0$.

Proposition 4.3 *The limiting statistics* $\rho_0(\tau)$ *can be computed as*

$$\rho_0(\tau) = - \inf_{x \in \mathbf{R}^m} \{ \ S_\tau(x) + V_\tau(x) \ \} \ ,$$

and the threshold lower bound $c_{\min}$ *as*

$$c_{\min} = - \inf_{0 \leq \tau \leq T'} \inf_{x \in \mathbf{R}^m} \{ \ S_\tau(x) + V_\tau(x) \ \} \ .$$

As an alternative to the exact computation of the threshold lower bound $c_{\min}$, it is interesting to see whether it would be possible to obtain estimates that would be easier to compute, and therefore could be used in practice.

□ Lower bound for $c_{\min}$

From the particular choice $\phi \equiv x^\tau$, it follows that

$$\rho_0(\tau) \geq -\tfrac{1}{2} \int_\tau^T |y_t - y_t^\tau|^2 \, dt \ .$$

From the particular choice $\phi \equiv x$, it follows that

$$\rho_0(\tau) \geq - [\ \tfrac{1}{2} \int_\tau^T |a(x_t)|^2 \, dt + \tfrac{1}{2} \int_\tau^T |g(x_t)|^2 \, dt \ ] \ ,$$

and therefore

$$c_{\min} \ \geq \ c''_{\min}$$

$$= - [\ \tfrac{1}{2} \int_{T'}^T |a(x_t)|^2 \, dt + \tfrac{1}{2} \int_{T'}^T |g(x_t)|^2 \, dt \ ] \ ,$$

where $c''_{\min}$ would be exactly the threshold lower bound in the case of complete observation, see [2].

□ Upper bound for $c_{\min}$

From the definition of $\rho_0(\tau)$, it follows that

$$\rho_0(\tau) \ \leq \ - \inf_\phi [\ \tfrac{1}{2} \int_\tau^T |\dot{\phi}_t - [b + a](\phi_t)|^2 \, dt$$

$$+ \tfrac{1}{2} \int_\tau^T |y_t - [h + g](\phi_t)|^2 \, dt \ ] \ ,$$

and therefore

$$c_{\min} \ \leq \ c'_{\min}$$

$$= - \inf_\phi [\ \tfrac{1}{2} \int_{T'}^T |\dot{\phi}_t - [b + a](\phi_t)|^2 \, dt$$

$$+ \tfrac{1}{2} \int_{T'}^T |y_t - [h + g](\phi_t)|^2 \, dt \ ] \leq 0 \ .$$

5 Probability of No Detection

If a change has occurred at time τ_0, the following limiting expression holds for the statistics $\rho_\varepsilon(\tau)$ under P_{ε,τ_0} as $\varepsilon \downarrow 0$

$$\begin{aligned}\rho(\tau_0,\tau) \;=\; & -\inf_{\phi\,:\,\phi_0=\bar{x}} \Big[\; \tfrac{1}{2}\int_0^T |\dot{\phi}_t - b^\tau(t,\phi_t)|^2\, dt \\ & \qquad + \tfrac{1}{2}\int_0^T |y_t^{\tau_0} - h^\tau(t,\phi_t)|^2\, dt \;\Big] \\ & + \inf_{\phi\,:\,\phi_0=\bar{x}} \Big[\; \tfrac{1}{2}\int_0^T |\dot{\phi}_t - b(\phi_t)|^2\, dt \\ & \qquad + \tfrac{1}{2}\int_0^T |y_t^{\tau_0} - h(\phi_t)|^2\, dt \;\Big] \;,\end{aligned}$$

where $\{y_t^{\tau_0}\,,\, t\geq 0\}$ is the output of the limiting deterministic system (2).

Proposition 5.1 *The following large deviations result holds*

$$P_{\varepsilon,\tau_0}\big(\sup_{0\leq\tau\leq T'} |\rho_\varepsilon(\tau) - \rho(\tau_0,\tau)| > \delta\big) \longrightarrow 0 \;,$$

when $\varepsilon\downarrow 0$.

It is clear that the mapping $\tau\mapsto\rho(\tau_0,\tau)$ achieves its maximum for $\tau=\tau_0$, which implies that the MLE for the estimation of the change time is consistent. Define

$$\rho^*(\tau_0) = \max_{0\leq\tau_0\leq T'} \rho(\tau_0,\tau) = \rho(\tau_0,\tau_0) \;,$$

i.e.

$$\begin{aligned}\rho^*(\tau_0) \;=\; & \inf_{\phi\,:\,\phi_0=\bar{x}} \Big[\; \tfrac{1}{2}\int_0^T |\dot{\phi}_t - b(\phi_t)|^2\, dt \\ & \qquad + \tfrac{1}{2}\int_0^T |y_t^{\tau_0} - h(\phi_t)|^2\, dt \;\Big] \geq 0 \;,\end{aligned}$$

where $\{y_t^{\tau_0}\,,\, t\geq 0\}$ is the output of the limiting deterministic system (2).

Proposition 5.2 *The probability of no detection M_ε goes to zero as $\varepsilon\downarrow 0$, provided the threshold c satisfies*

$$c < c_{\max} = \inf_{0\leq\tau_0\leq T'} \rho^*(\tau_0) \;. \tag{4}$$

The upper bound $c_{\max}$ can be computed exactly, using a family, parametrized by the *true* change time τ_0, of dynamic programming equations, all of them involving the coefficients before the change, but using different outputs $\{y_t^{\tau_0}\,,\, 0\leq t\leq T\}$. With this formulation, optimization over the path space Ω is repaced by optimization over the state space $\mathbf{R}^m$.

Indeed, consider the forward dynamic programming equation

$$\frac{\partial S_t^{\tau_0}}{\partial t} = -\tfrac{1}{2}\,|\nabla S_t^{\tau_0}|^2 - b\,\nabla S_t^{\tau_0} + \tfrac{1}{2}\,|y_t^{\tau_0} - h|^2 \;,$$

with initial condition $S_0^{\tau_0}(\bar{x}) = 0$, and $S_0^{\tau_0}(x) = \bar{S}$ (with $\bar{S}$ large enough) for $x\neq\bar{x}$.

Proposition 5.3 *The limiting statistics $\rho^*(\tau_0)$ can be computed as*

$$\rho^*(\tau_0) = \inf_{x\in\mathbf{R}^m} S_T^{\tau_0}(x) \;,$$

and the threshold upper bound $c_{\max}$ as

$$c_{\max} = \inf_{0\leq\tau_0\leq T'}\; \inf_{x\in\mathbf{R}^m} S_T^{\tau_0}(x) \;.$$

Here again, as an alternative to the exact computation of the threshold upper bound $c_{\max}$, it is interesting to see whether it would be possible to obtain estimates that would be easier to compute, and therefore could be used in practice.

□ Upper bound for $c_{\max}$

From the particular choice $\phi\equiv x$, it follows that

$$\rho^*(\tau_0) \leq \tfrac{1}{2}\int_{\tau_0}^T |y_t - y_t^{\tau_0}|^2\, dt \;.$$

From the particular choice $\phi\equiv x^{\tau_0}$, it follows that

$$\rho^*(\tau_0) \leq \tfrac{1}{2}\int_{\tau_0}^T |a(x_t^{\tau_0})|^2\, dt + \tfrac{1}{2}\int_{\tau_0}^T |g(x_t^{\tau_0})|^2\, dt \;,$$

and therefore

$$\begin{aligned} c_{\max} \;\leq\; & c''_{\max} \\ =\; & \inf_{0\leq\tau_0\leq T'} \Big[\; \tfrac{1}{2}\int_{\tau_0}^T |a(x_t^{\tau_0})|^2\, dt \\ & \qquad + \tfrac{1}{2}\int_{\tau_0}^T |g(x_t^{\tau_0})|^2\, dt \;\Big] \;,\end{aligned}$$

where $c''_{\max}$ would be exactly the threshold upper bound in the case of complete observation, see [2].

□ Lower bound for $c_{\max}$

From the definition of $\rho^*(\tau_0)$, it follows that

$$\begin{aligned}\rho^*(\tau_0) \;\geq\; & \inf_{\phi} \Big[\; \tfrac{1}{2}\int_{\tau_0}^T |\dot{\phi}_t - b(\phi_t)|^2\, dt \\ & \qquad + \tfrac{1}{2}\int_{\tau_0}^T |y_t^{\tau_0} - h(\phi_t)|^2\, dt \;\Big] \;,\end{aligned}$$

and therefore

$$
\begin{aligned}
c_{\max} \; &\geq \; c'_{\max} \\
&= \inf_{0 \leq \tau_0 \leq T'} \; \inf_{\phi} \Big[\; \tfrac{1}{2} \int_{\tau_0}^{T} |\dot{\phi}_t - b(\phi_t)|^2 \, dt \\
&\qquad\qquad + \tfrac{1}{2} \int_{\tau_0}^{T} |y_t^{\tau_0} - h(\phi_t)|^2 \, dt \; \Big] \geq 0 \; .
\end{aligned}
$$

6 Conclusion

From the previous discussion, it is possible to select the threshold c in such a way that both the probability of false alarm and the probability of no detection go to zero as $\varepsilon \downarrow 0$. Actually, the threshold c should satisfy simultaneously

$$c_{\min} < c < c_{\max} \; ,$$

where the threshold bounds $c_{\min} \leq 0$ and $c_{\max} \geq 0$ have been defined in (3) and (4) respectively. This is possible provided the following *detectability* assumption holds

$$c_{\min} < c_{\max} \; , \tag{5}$$

i.e. whenever one of the two above quantities is non–zero. Remark that for this assumption to hold, it is necessary that $T' < T$.

Estimates

$$c''_{\min} \leq c_{\min} \leq c'_{\min} \leq 0 \; ,$$

and

$$0 \leq c'_{\max} \leq c_{\max} \leq c''_{\max} \; ,$$

have also been obtained, where $c''_{\min}$ and $c''_{\max}$ would be the threshold bounds in the case of complete observation. Therefore, a *sufficient* condition for (5) to hold, is

$$c'_{\min} < c'_{\max} \; ,$$

whereas a *necessary* condition would be

$$c''_{\min} < c''_{\max} \; .$$

Robustness w.r.t. mis–specification of the change coefficients $a(\cdot)$ and $g(\cdot)$ can also be investigated, as in [2].

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