Filtering by Projection on the Manifold of Exponential Densities

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VRIJE UNIVERSITEIT

FILTERING BY PROJECTION ON THE MANIFOLD OF EXPONENTIAL DENSITIES

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Vrije Universiteit te Amsterdam,
op gezag van de rector magnificus
prof.dr. E. Boeker,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de faculteit der economische wetenschappen en econometrie
op dinsdag 29 oktober 1996 te 15.45 uur
in het hoofdgebouw van de universiteit,
De Boelelaan 1105

door

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geboren te Venetië, Italië

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Funding

This project was mainly supported by the following Italian institutions:

- Università degli studi di Padova,
- Istituto Nazionale di Alta Matematica 'F. Severi',

and later on by

- The European Union, with a fellowship of the Training and Mobility of Researchers programme, under contract number FMBI–CT96–0791.

Further funding was provided by:

- The Dutch Systems and Control Theory Network (University of Groningen),
- The European Union, with the Networks of the SCIENCE Project *System Identification* (contract number SC1*–CT92–0779), and the HCM Network *Statistical Inference for Stochastic Processes* (contract number CHRX–CT92–0078),
- The Army Research Office, under grant DAAH04–95–1–0164,
- The Centrum voor Wiskunde en Informatica (CWI) in Amsterdam,
- The Department of Economics and Econometrics of the Free University of Amsterdam.

Summaries

French summary written with the help of Marc Joannides,
Dutch summary written with the help of Bernard Hanzon.

Alla mia famiglia

Girovago

Campo di Mailly maggio 1918

In nessuna
parte
di terra
mi posso
accasare

A ogni
nuovo
clima
che incontro
mi trovo
languente
che
una volta
già gli ero stato
assuefatto

E me ne stacco sempre
straniero

Nascendo
tornato da epoche troppo
vissute

Godere un solo
minuto di vita
iniziale

Cerco un paese
innocente

Giuseppe Ungaretti

Blessed are the poor in spirit, for theirs is the kingdom of heaven.
Blessed are those who mourn, for they will be comforted.
Blessed are the meek, for they will inherit the earth.
Blessed are those who hunger and thirst for righteousness,
for they will be filled.
Blessed are the merciful, for they will be shown mercy.
Blessed are the pure in heart, for they will see God.
Blessed are the peacemakers, for they will be called sons of God.
Blessed are those who are persecuted because of righteousness,
for theirs is the kingdom of heaven.

Matthew V.3–10

How sure his pathway in this wood,
Who follows truth's unchanging call!
How blessed, to be kind and good,
And practice self–restraint in all!
How light, from passion to be free,
And sensual joys to let go by!
And yet his greatest bliss will be
When he has quelled the pride of 'I'

Paul Carus, from the Gospel of Buddha.

Contents

Chapter 1

Introduction

It looked insanely complicated, and this was one of the reasons why the snug plastic cover it fitted into had the words DON'T PANIC printed on it in large friendly letters.

D. Adams, The Hitch–Hikers Guide to the Galaxy.

1.1 General introduction

The present thesis concerns the finite–dimensional approximation of distributions obtained via differential–geometric methods and exponential families. We treat mainly the nonlinear filtering problem, although some results on diffusion processes and stochastic differential equations (SDE's) are given. In the present introductory chapter we describe the filtering problem, our approach to its solution, the particular role that exponential families and differential geometry play in our method, and our results. We present some hints about possible and potential applications of our results. This introductory chapter is written in very general terms, and is meant to be readable by mathematicians, econometricians and engineers who are not necessarily probabilists.

The filtering problem

The filtering problem consists of estimating the state of a stochastic system from noise perturbed observations. One has a system whose state evolves according to a stochastic (difference or differential) equation, and one observes a related process which is generally a function of the state process plus a disturbance. This function is not bijective in general, so that it cannot be inverted to recover the state (even in the case where no disturbance is present in the observations). This is usually referred to as the case of *partial observations.* The filtering problem consists of estimating the signal at any time instant from the history of the observation process up to the same instant. As a simple example, consider a population growth model. Let $X(t)$ be the population at time $t \in \mathbf{R},\ t \geq 0$. The simplest model for the population growth is obtained by assuming that the growth rate is proportional to the current population. This can be translated into the ordinary differential equation:

$$\dot{X}(t) = kX(t), \quad X(0) = X_0, \quad k \geq 0.$$

Now suppose that, due to some complications, it is no longer realistic to assume both k and X_0 to be deterministic constants. Then we may decide to let X_0 be a random variable, and to model the population growth by taking in account a noise process $\{v_1(t),\ t \geq 0\}$, so that

$$\dot{X}(t) = (k + v_1(t))\ X(t), \quad X(0) = X_0, \quad k \geq 0.$$

Now assume that we cannot observe X but, due to some limitations in the observation procedure, we can observe only a disturbed measure of $X(t)$:

$$Y(t) = X(t) + v_2(t),$$

where $\{v_2(t),\ t \geq 0\}$ is a second noise process. The filtering problem, in this example, consists of estimating $X(t)$ on the basis of $\{Y(s): \quad 0 \leq s \leq t\}$. In general, in our framework, the state X will evolve according to a stochastic differential equation describing what is known as a diffusion process, which has a structure more complicated than in the example above. The observations have also a more complicated structure in general.

If the evolution of the state X and the observations Y are described by linear equations, under some assumptions which completely specify the probabilistic behaviour of the initial condition X_0, the solution of the problem is the well known Kalman Filter. This filter consists of a *finite set* of recursive equations which permit to update the estimates including at each time instant the new observations (more precisely, if the system evolves in discrete time we have a finite set of difference equations, whereas in continuous time we have a finite set of differential equations). In this case the optimal filter is termed *finite dimensional.* Although the Kalman filter works only in the context of linear systems, it turned out to be very useful in many applications. In the past it was used for example in aerospace applications (Ranger, Mariner, Apollo), in water–level prediction and in underwater applications. Currently, the linear filter is applied in many fields of engineering and economics. Still, as Michiel

Hazewinkel wrote in the editor's preface for the series *Mathematics and Its Applications* published by Kluwer Academic :

> *In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.*

The more general nonlinear filtering problem is far more complicated because the resulting nonlinear filter is not finite dimensional in general. Finite dimensionality of a filter is loosely defined as a filter consisting of a finite set of recursive equations which update the conditional distribution of the state based on the past observations. In general there is no such finite set of equations for the nonlinear filtering problem. In such a case the solution cannot be implemented by a computer with finite memory. The solution of the filtering problem in continuous time is a conditional density which is described by a mathematical object called a stochastic partial–differential equation. This is in general an infinite–dimensional equation, in the sense that its solution cannot be characterized by the solution of a finite set of (stochastic) differential equations. A well known approximation method to find a remedy to this infinite dimensionality is the extended Kalman filter (EKF). The EKF is obtained by linearizing the equations for X and Y around the current estimates and by applying the linear filter. This procedure is usually justified on the basis of heuristic considerations, and not much is known about the quality of its performances, except in the case of small observation noise. Another choice in the nonlinear case is what is known as Gaussian assumed–density filter (GADF). Roughly speaking, the optimal filter can be 'characterized' by an infinite number of parameters. Now, it is possible to privilege a finite number of these parameters and to ignore the others so as to obtain a finite set of recursive equations describing the evolution of the privileged parameters. In other words, one arbitrarily assumes the infinite dimensional filter to be characterized by the privileged parameters. This procedure produces a finite dimensional filter. Still, this is very dangerous from a mathematical standpoint, because from a false hypothesis no interesting scientific statements can be deduced.

Our approach

We present a new way to obtain a finite set of (stochastic) differential equations which approximate the infinite–dimensional equation for the optimal filter. The projection filter (PF) is a finite–dimensional nonlinear filter based on the differential–geometric approach to statistics. By using geometry, we construct a procedure to project the infinite–dimensional equation for the optimal filter onto a finite–dimensional space. This projection is mathematically well defined. Moreover, there is ample choice about what finite–dimensional space one can project upon.

Exponential families

In this thesis we use this geometric framework to define and study in detail the projection filter for *exponential families*. The choice of exponential families is somehow natural, since they simplify both the obtained equations and the conditions under which such equations admit solutions. Moreover, we need exponential families in order to be able to define a single quantity (called *total projection residual*) which measures the local approximation involved in the projection around each time instant. For general parametric families we do not know how to define such a quantity. We also prove that the projection filter is equivalent to the assumed–density filter based on the McShane-Fisk-Stratonovich (MFS) representation and exponential families. This equivalence holds only for exponential families. Simulations have been performed for the exponential projection filter applied to a particular system called *cubic sensor*. Finally, some results on the nice asymptotic behaviour of the Gaussian projection filter with small observation noise are given. The Gaussian densities are a particular case of exponential densities.

Potential applications

The theory developed so far has only be tested on an academic example, the cubic sensor. Applications in economics and engineering are under study. We are currently investigating the possible use of projection filters for estimating the volatility of bilateral exchange rates, in the context of applications to mathematical finance. The first results in this direction can be found in [12].

Applications of nonlinear filtering require numerical tools and software. We are planning to work on this in order to render the projection filter a possible tool for solving concrete filtering problems in engineering, economics and other applied fields.

Stochastic differential equations with densities evolving in exponential families

The theory developed for the filtering problem shows some potential for other applications. In the thesis we solve several problems related to the existence of stochastic differential equations the density of which evolves in a specified exponential family.

In the above presentation we described the contribution of our work in a very informal way. In the following we explain how to use this thesis and we describe its contents chapter by chapter. This will require a more specific language.

1.2 How to use this thesis (strongly recommended reading)

If you are interested only in the key ideas behind the projection in Fisher metric of an equation describing the evolution of a density (for example the Fokker–Planck equation) and you do not want to go through too many theoretical details, the beginning of Chapter 4 can be helpful. Then you may skip the derivation of the projection filter and just read Theorem 4.4.3 in order to see the filter equations.

If you wish to have a quick intuitive idea of the key elements involved in projecting the nonlinear filtering equation onto a finite–dimensional manifold of densities you can read Section 6.6 of Chapter 6 where we treat a particular case from an intuitive point of view. You can also read Chapter 7 where we project the Fokker–Planck equation onto an exponential manifold of densities. The Fokker–Planck equation corresponds to the prediction step in filtering, and it is this step that brings in infinite dimensionality, since the update step can be handled exactly (see Section 4.5.2, Chapter 4). Moreover, the simpler structure of the Fokker–Planck equation can help in grasping the key ideas of the projection technique without being overwhelmed by notation and details, as could happen in the case of the complete nonlinear–filtering equation.

If you are not familiar with geometric concepts and would like to have a different point of view on the projection filter, you can read Chapter 5, where we give a characterization of the projection filter based on the assumed–density principle (which is not intrinsically based on geometrical concepts).

If you are interested on the small observation noise approach, our results concern the Gaussian projection filter which can be derived via the assumed–density principle. We give an independent derivation of this filter in the chapter on small noise in order to keep the chapter as self contained as possible. There are very little geometric facts in that chapter, so (again) if you are not familiar with geometry it can be a good reading.

For the reader interested in the whole derivation and in the whole theory, the thesis runs as follows.

1.3 Short description of contents by chapter

Chapter 2 introduces quickly some elements needed from the theory of stochastic differential equations and the filtering problem. We refer to the literature for the presented results.

Chapter 3 introduces the theory of statistical manifolds, i.e. how to install a geometric structure in a parametrized family of probability densities. Actually, no advanced differential–geometric tools are needed. The important geometric concepts are: tangent vectors, projections, and Riemannian metrics. We consider the L_2 metric (Hellinger distance) and show that it coincides with the well known Fisher information metric. We particularize the theory to exponential families and give some known results on them.

Chapter 4 introduces and studies the projection filter. We start by developing some intuition on the projection in Fisher metric of an equation describing the evolution of a density by treating informally the projection in Fisher metric of the Fokker–Planck equation onto a finite–dimensional manifold of densities. Next, we introduce rigorously the exponential projection filter. In the construction of the geometrical framework we use an enveloping manifold for the stochastic partial differential equation of the optimal filter. This manifold framework will be useful in proving equivalence with the assumed–density filter that will be formulated in the following chapter, otherwise one could use only the L_2 structure without worrying about the enveloping manifold. We prove existence of the projection filter for exponential families of densities and define the projection residuals, which are quantities meant to measure the local quality of the approximation involved in using the projection filter. We introduce some particular manifolds defined in terms of the given filtering problem. Such manifolds allow the simplification of the projection–filter equation and of the projection residuals. Finally, we apply the developed theory to the cubic–sensor problem and develop numerical simulations where we compare the resulting projection filter with the optimal filter obtained via local–grid approximation.

In Chapter 5 we prove equivalence between projection filters and assumed–density filters (based on Stratonovich's calculus) for exponential families. This will yield both a non–geometrical characterization of the projection filter and logical consistency for the assumed–density filter, which is otherwise based on logical inconsistency. The equivalence is limited to assumed–density filters with an exponential family, anyway, as we show with an example. Moreover, the assumed–density filter based on Itô's calculus will be proven to be different from the corresponding Stratonovich–based one.

In Chapter 6 we treat the Gaussian projection filter with small observation noise. The chapter is as self-contained as possible. We derive the Gaussian projection filter via the assumed–density approach. Then we prove that the L_2 distance between the true state and the estimated state of the two–dimensional Gaussian projection filter is at most proportional to the observation noise. We show that the same property holds for a one–dimensional Gaussian projection filter. We give results for different type of problems and finally we show that the L_2 distance between the optimal filter and the Gaussian projection filter is at most proportional to the *square* of the observation noise.

Chapter 7 gives results concerning stochastic differential equations whose density evolves in a finite–dimensional family. This chapter is intended to show how the projection in Fisher metric can be a useful tool not only in the filtering theory but also in the broader field of stochastic differential equations. More specifically, we show that the density evolution obtained from the projection in Fisher metric of the density evolution of a diffusion process onto an exponential manifold can be interpreted as the density evolution of a different diffusion sharing the same diffusion coefficient. We show that, given a priori a diffusion coefficient and an exponential family, we can define a drift such that the density evolution of the related diffusion remains in the exponential manifold. We apply this result to the construction of nonlinear SDE's with prescribed exponential–density evolution and in particular with prescribed stationary exponential density. We illustrate the usefulness of this result with an example from mathematical finance. We present a stability result for the projected density evolution of some particular diffusions. Finally, we apply these results to the construction of some nonlinear–filtering problems for which the optimal filter is finite dimensional. Some of the coefficients of the problem can be assigned a priori and we can give the remaining coefficient in such a way that the optimal filter for the resulting problem be finite dimensional.

Material appeared in publications

Many of the results presented here have already appeared as reports or publications. Here we give a short list of these publications. The IRISA report [13] contains (with some minor differences) material from Chapters 2, 3, and a short part of Chapter 5.

The proceedings [14] contain a short version of this report, and a modified version of the proceedings was recently revised and resubmitted for publication in *IEEE Transactions on Automatic Control*.

The Tinbergen–Institute report [16] contains material from Chapters 2, 3, a part from Chapter 4, and the whole Chapter 5.

The two papers in *Systems & Control Letters*, [6], [8], and the proceedings [7] contain the results of Chapter 6, apart from the comparison optimal filter–Gaussian projection filter, which has been submitted recently for publication (see [10]).

The LADSEB–CNR report [9] contains results that have been perfected in Chapter 7 of the present thesis. The conference version [11] is also related to the material presented in [9] and in Chapter 7.

The report [17] contains results related to Chapter 7, derived in the deeper geometrical framework of Pistone and Sempi [54].

The work [15] was recently submitted to *Bernoulli*. It is intended to be the main article on the projection filter. It contains material from the Chapters 2, 3, 4, 5. The simulations of Chapter 4 are not included in this article.

The working paper [12] concerns projection filters in discrete time and has not been included in the present thesis, which is devoted to continuous–time models.

Chapter 2

Stochastic Differential Equations and the Filtering Problem

You have not played as yet? Do not do so.
Above all, avoid a martingale if you do.
W. M. Thackeray

2.1 Introduction

In this chapter we introduce the nonlinear filtering problem. The filtering problem concerns the possibility of estimating the state of a dynamical system from the past and current observations of a related measurement process. A stochastic system is given, whose state evolves according to a stochastic differential equation (SDE). The problem consists for example of estimating the state from nonlinear observations in additive Gaussian white noise. In the linear Gaussian case the solution consists of the Kalman filter, a finite–dimensional algorithm which computes the first two conditional moments of the state given the observations. Such an algorithm provides also the complete conditional density of the state given the observations, since in the linear case this conditional density is Gaussian and hence characterized by the first two moments. From a probabilistic point of view, the filtering problem consists in the calculation of the whole conditional density, which as a rule results in an infinite dimensional filter in the general nonlinear case. Under some regularity conditions, the conditional density exists and is the solution of the Kushner–Stratonovich equation, a stochastic partial differential equation.

For an introduction to stochastic calculus and to SDE's we refer to the book of Karatzas and Shreve [36]. For an extended introduction to the filtering problem we refer to the following articles: the tutorial of Davis and Marcus [19], and the article of van Schuppen [57]. Books on filtering are also available: for a treatment of the filtering problem from a mathematical point of view see the book of Liptser and Shiryayev [45], or the book of Kallianpur [34]. For books on filtering from a more applied perspective see the book of Jazwinski [32] or Maybeck [47].

2.2 The nonlinear filtering problem

On the probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\,,\, t \geq 0\}$ we consider the following state and observation equations, see Jazwinski [32], Maybeck [47], Davis and Marcus [19] :

$$\begin{aligned} dX_t &= f_t(X_t)\,dt + \sigma_t(X_t)\,dW_t, \quad X_0, \\ dY_t &= h_t(X_t)\,dt + dV_t, \quad Y_0 = 0\ . \end{aligned} \tag{2.1}$$

These equations are Itô stochastic differential equations (SDE's). In this thesis we shall use both Itô SDE's (for example for the signal X) and McShane–Fisk–Stratonovich (MFS) SDE's (when dealing with manifolds and projections). The MFS form will be denoted by the presence of the symbol '$\circ$' in between the diffusion coefficient and the Brownian motion of a SDE. The use of MFS SDE's is necessary in order to be able to deal with stochastic calculus on manifolds, since in general one does not know how to interpret the second order terms arising in Itô's calculus in terms of manifold structures. The interested reader is referred to [22].

In (2.1), the unobserved state process $\{X_t\,,\, t \geq 0\}$ and the observation process $\{Y_t\,,\, t \geq 0\}$ are taking values in $\mathbf{R}^n$ and $\mathbf{R}^d$ respectively, the noise processes $\{W_t\,,\, t \geq 0\}$ and $\{V_t\,,\, t \geq 0\}$ are two Brownian motions, taking values in $\mathbf{R}^p$ and $\mathbf{R}^d$ respectively, with covariance matrices Q_t and R_t respectively. We assume that R_t is invertible for all $t \geq 0$, which implies that, without loss of generality, we can assume that $R_t = I$ for all $t \geq 0$. Finally, the initial state X_0 and the noise processes $\{W_t\,,\, t \geq 0\}$ and $\{V_t\,,\, t \geq 0\}$ are assumed to be independent. We assume that the initial state X_0 has a density p_0 w.r.t. the Lebesgue measure λ on $\mathbf{R}^n$, and has finite moments of any order, and we make the following assumptions on the coefficients f_t, $a_t := \sigma_t\, Q_t\, \sigma_t^T$, and h_t of the system (2.1)

(A) Local Lipschitz continuity : for all $R > 0$, there exists $K_R > 0$ such that

$$|f_t(x) - f_t(x')| \leq K_R\, |x - x'| \qquad \text{and} \qquad \|a_t(x) - a_t(x')\| \leq K_R\, |x - x'| ,$$

for all $t \geq 0$, and for all $x, x' \in B_R$, the ball of radius R.

(B) Non–explosion : there exists $K > 0$ such that

$$x^T f_t(x) \leq K\, (1 + |x|^2) \qquad \text{and} \qquad \text{trace } a_t(x) \leq K\, (1 + |x|^2) ,$$

for all $t \geq 0$, and for all $x \in \mathbf{R}^n$.

(C) Polynomial growth : there exist $K > 0$ and $r \geq 0$ such that

$$|h_t(x)| \leq K\, (1 + |x|^r) ,$$

for all $t \geq 0$, and for all $x \in \mathbf{R}^n$.

Under assumptions (A) and (B), there exists a unique solution $\{X_t\, , \, t \geq 0\}$ to the state equation, see [37] or [36], and X_t has finite moments of any order. Under the additional assumption (C) the following *finite energy* condition holds

$$E \int_0^T |h_t(X_t)|^2 \, dt < \infty , \qquad \text{for all } T \geq 0.$$

The nonlinear filtering problem consists in finding the conditional probability distribution π_t of the state X_t given the observations up to time t, i.e. $\pi_t(dx) := P[X_t \in dx \mid \mathcal{Y}_t]$, where $\mathcal{Y}_t := \sigma(Y_s\, , \, 0 \leq s \leq t)$. Since the finite energy condition holds, it follows from Fujisaki, Kallianpur and Kunita [27] that $\{\pi_t\, , \, t \geq 0\}$ satisfies the Kushner–Stratonovich equation, i.e. for any smooth and compactly supported test function ϕ defined on $\mathbf{R}^n$

$$\pi_t(\phi) = \pi_0(\phi) + \int_0^t \pi_s(\mathcal{L}_s \phi)\, ds + \sum_{k=1}^d \int_0^t [\pi_s(h_s^k\, \phi) - \pi_s(h_s^k)\, \pi_s(\phi)]\, [dY_s^k - \pi_s(h_s^k)\, ds] , \tag{2.2}$$

where for all $t \geq 0$, the backward diffusion operator $\mathcal{L}_t$ is defined by

$$\mathcal{L}_t = \sum_{i=1}^n f_t^i \, \frac{\partial}{\partial x_i} + \tfrac{1}{2} \sum_{i,j=1}^n a_t^{ij} \, \frac{\partial^2}{\partial x_i \partial x_j} .$$

The MFS form of equation (2.2) is obtained, after straightforward computations, as :

$$\begin{aligned} \pi_t(\phi) \;&=\; \pi_0(\phi) + \int_0^t \pi_s(\mathcal{L}_s\, \phi)\, ds - \tfrac{1}{2} \int_0^t [\pi_s(|h_s|^2\, \phi) - \pi_s(|h_s|^2)\, \pi_s(\phi)]\, ds \\ &\qquad + \sum_{k=1}^d \int_0^t [\pi_s(h_s^k\, \phi) - \pi_s(h_s^k)\, \pi_s(\phi)] \circ dY_s^k . \end{aligned} \tag{2.3}$$

From now on we proceed formally, and we assume that for all $t \geq 0$, the probability distribution π_t has a density p_t w.r.t. the Lebesgue measure on $\mathbf{R}^n$. Then $\{p_t \, , \, t \geq 0\}$ satisfies the Itô–type stochastic partial differential equation (SPDE)

$$dp_t = \mathcal{L}_t^* \, p_t \, dt + \sum_{k=1}^{d} p_t \, [h_t^k - E_{p_t}\{h_t^k\}] \, [dY_t^k - E_{p_t}\{h_t^k\} \, dt] \tag{2.4}$$

in a suitable functional space, where $E_{p_t}\{\cdot\}$ denotes the expectation w.r.t. the probability density p_t, i.e. the conditional expectation given the observations up to time t, and where for all $t \geq 0$, the forward diffusion operator $\mathcal{L}_t^*$ is defined by

$$\mathcal{L}_t^* \phi = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [f_t^i \, \phi] + \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [a_t^{ij} \, \phi] \ ,$$

for any test function ϕ defined on $\mathbf{R}^n$. The corresponding MFS form of the SPDE (2.4) is :

$$dp_t = \mathcal{L}_t^* \, p_t \, dt - \tfrac{1}{2} \, p_t \, [|h_t|^2 - E_{p_t}\{|h_t|^2\}] \, dt + \sum_{k=1}^{d} p_t \, [h_t^k - E_{p_t}\{h_t^k\}] \circ dY_t^k \ .$$

In order to simplify notation, we introduce the following definitions, which will be used throughout the following chapters :

$$\begin{aligned} \alpha_t(p) := \frac{\mathcal{L}_t^* \, p}{p} \ , \qquad \beta_t^0(p) \ &:= \ \tfrac{1}{2} \, [|h_t|^2 - E_p\{|h_t|^2\}] \ , \\ \beta_t^k(p) \ &:= \ h_t^k - E_p\{h_t^k\} \ , \end{aligned} \tag{2.5}$$

for $k = 1, \cdots, d$. Simple calculations show that

$$\begin{aligned} \alpha_t(p) = &- \sum_{i=1}^{n} [\, f_t^i \, \frac{\partial}{\partial x_i} (\log p) + \frac{\partial f_t^i}{\partial x_i} \,] \\ &+ \tfrac{1}{2} \sum_{i,j=1}^{n} [\, a_t^{ij} \, \frac{\partial^2}{\partial x_i \partial x_j} (\log p) + a_t^{ij} \, \frac{\partial}{\partial x_i} (\log p) \, \frac{\partial}{\partial x_j} (\log p) \\ &\qquad + 2 \, \frac{\partial a_t^{ij}}{\partial x_j} \, \frac{\partial}{\partial x_i} (\log p) + \frac{\partial^2 a_t^{ij}}{\partial x_i \partial x_j} \,] \ . \end{aligned} \tag{2.6}$$

The MFS form of the Kushner–Stratonovich equation reads now

$$dp_t = \mathcal{L}_t^* \, p_t \, dt - p_t \, \beta_t^0(p_t) \, dt + \sum_{k=1}^{d} p_t \, \beta_t^k(p_t) \circ dY_t^k \ .$$

We shall frequently work with square roots of densities, rather than densities themselves. Then, we compute by formal rules, using the MFS form :

$$
\begin{aligned}
d\sqrt{p_t} = \frac{1}{2\sqrt{p_t}} \circ dp_t \quad &= \quad \tfrac{1}{2}\sqrt{p_t}\,\alpha_t(p_t)\,dt - \tfrac{1}{2}\sqrt{p_t}\,\beta_t^0(p_t)\,dt \\
&\quad + \tfrac{1}{2}\textstyle\sum_{k=1}^d \sqrt{p_t}\,\beta_t^k(p_t) \circ dY_t^k \\
&= \quad \mathcal{P}_t(\sqrt{p_t})\,dt - \mathcal{Q}_t^0(\sqrt{p_t})\,dt + \sum_{k=1}^d \mathcal{Q}_t^k(\sqrt{p_t}) \circ dY_t^k \ ,
\end{aligned}
\tag{2.7}
$$

where the nonlinear time dependent operators $\mathcal{P}_t$ and $\mathcal{Q}_t^k$ for $k = 0, 1, \cdots, d$ are defined by

$$
\mathcal{P}_t(r) := \tfrac{1}{2}\, r\, \alpha_t(r^2) = \frac{\mathcal{L}_t^*\, r^2}{2r} \ , \qquad \mathcal{Q}_t^k(r) := \tfrac{1}{2}\, r\, \beta_t^k(r^2) \tag{2.8}
$$

respectively.

Chapter 3

Statistical Manifolds and Fisher Information

$\mathcal{DON'T} \quad \mathcal{PANIC}$

The Hitch–Hikers Guide to the Galaxy

3.1 Statistical manifolds

On the measurable space $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ we consider a non–negative and σ–finite measure λ, and we define $\mathcal{M}(\lambda)$ to be the set of all non–negative and finite measures μ which are absolutely continuous w.r.t. λ, and whose density

$$p_\mu = \frac{d\mu}{d\lambda}$$

is positive λ–a.e. For simplicity, we restrict ourselves in this thesis to the case where λ is the Lebesgue measure on $\mathbf{R}^n$.

In the following, we denote by $H(\lambda)$ the set of all the densities of measures contained in $\mathcal{M}(\lambda)$. Notice that, as all the measures in $\mathcal{M}(\lambda)$ are non–negative and finite, we have that if p is a density in $H(\lambda)$ then $p \in L_1(\lambda)$, that is $\sqrt{p} \in L_2(\lambda)$. The above remark implies that the set $\mathcal{R}(\lambda) := \{\sqrt{p} : p{\in}H(\lambda)\}$ of square roots of densities of $H(\lambda)$ is a subset of $L_2(\lambda)$. Notice that all $\sqrt{p}$ in $\mathcal{R}(\lambda)$ satisfy $\sqrt{p(x)} > 0$, for almost every $x \in \mathbf{R}^n$. The above remarks lead to the definition of the following metric in $\mathcal{R}(\lambda)$, see Jacod and Shiryayev [31] or Hanzon [28] : $d_{\mathcal{R}}(\sqrt{p_1}, \sqrt{p_2}) := \|\sqrt{p_1} - \sqrt{p_2}\|$, where $\|\cdot\|$ denotes the norm of the Hilbert space $L_2(\lambda)$. This leads to the Hellinger metric on $H(\lambda)$ (or $\mathcal{M}(\lambda)$), obtained by using the bijection between densities (or measures) and square roots of densities : if μ_1 and μ_2 are the measures having densities p_1

and p_2 w.r.t. λ, the Hellinger metric is defined as $d_{\mathcal{M}}(\mu_1, \mu_2) = d_H(p_1, p_2) = d_{\mathcal{R}}(\sqrt{p_1}, \sqrt{p_2})$. It can be shown, see e.g. [28], that the distance $d_{\mathcal{M}}(\mu_1, \mu_2)$ in $\mathcal{M}(\lambda)$ is defined independently of the particular λ we choose as basic measure, as long as both μ_1 and μ_2 are absolutely continuous w.r.t. λ. As one can always find a λ such that both μ_1 and μ_2 are absolutely continuous w.r.t. λ (take for example $\lambda := (\mu_1 + \mu_2)/2$), the distance is well defined on the set of all finite and positive measures on $(\Omega, \mathcal{F})$. Notice that $\mathcal{R}(\lambda)$ is not locally homeomorphic to $L_2(\lambda)$, hence is not a manifold modeled on $L_2(\lambda)$. Indeed, any open set of $L_2(\lambda)$ contains functions which are negative in a set with positive λ–measure. There is no open set of $L_2(\lambda)$which contains only positive functions such as the functions of $L_2(\lambda)$.

In the following we give a very quick review of the main concepts we need from differential geometry. For a survey on the role of differential geometry in statistical theory see for example [4]. For the basic definitions and a more technical introduction on manifolds, tangent vectors and related concepts we refer to the literature, see for example [2] and the references given therein, and [43].Consider an open subset M of $L_2(\lambda)$. Let x be a point of M, and let $\gamma : (-\epsilon, \epsilon) \rightarrow M$ be a curve on M around x, i.e. a differentiable map between an open neighborhood of $0 \in \mathbf{R}$ and M such that $\gamma(0) = x$. We can define the tangent vector to γ at x as the Fréchet derivative $D\gamma(0) : (-\epsilon, \epsilon) \rightarrow L_2(\lambda)$, i.e. the linear map defined in $\mathbf{R}$ around 0 and taking values in $L_2(\lambda)$ such that the following limit holds :

$$\lim_{|h| \to 0} \frac{\|\gamma(h) - \gamma(0) - D\gamma(0) \cdot h\|}{|h|} = 0 \ .$$

The map $D\gamma(0)$ approximates linearly the change of γ around x. Let $\mathcal{C}_x(M)$ be the set of all the curves on M around x. If we consider the space

$$L_x M := \{D\gamma(0) \, : \, \gamma \in \mathcal{C}_x(M)\} \ ,$$

of tangent vectors to all the possible curves on M around x, we obtain again the space $L_2(\lambda)$. This is due to the fact that for every $v \in L_2(\lambda)$ we can always consider the straight line $\gamma^v(h) := x + h\, v$. Since M is open, $\gamma^v(h)$ takes values in M for $|h|$ small enough. Of course $D\gamma^v(0) = v$, so that indeed $L_x M = L_2(\lambda)$. The situation becomes different if we consider an m–*dimensional* manifold N imbedded in $L_2(\lambda)$. We can consider the induced L_2 structure on N as follows : suppose $x \in N$, and define again

$$L_x N := \{D\gamma(0) \, : \, \gamma \in \mathcal{C}_x(N)\} \ .$$

This is a linear subspace of $L_2(\lambda)$ called the *tangent vector space* at x, which does not coincide with $L_2(\lambda)$ in general (due to the finite dimension of N).

The set of all tangent vectors at all points x of N is called the *tangent bundle*, and will be denoted by LN. In our work we shall consider finite dimensional manifolds N embedded in $L_2(\lambda)$, which are contained in $\mathcal{R}(\lambda)$ as a set, i.e. $N \subset \mathcal{R}(\lambda) \subset L_2(\lambda)$, so that usually $x = \sqrt{p}$. It may be important to point out that, although we are using square roots of densities in order to keep the L_2 structure, once we have a finite dimensional manifold N, we can consider any of the embeddings $\sqrt{p} \mapsto \mu_p$, or $\sqrt{p} \mapsto p$, focusing on manifolds of probability measures μ_p, or their densities p rather than on their square roots $\sqrt{p}$.

If N is m–dimensional, it is locally homeomorphic to $\mathbf{R}^m$, and it may be described locally by a chart : if $\sqrt{p} \in N$, there exists a pair $(S^{1/2}, \phi)$ with $S^{1/2}$ open neighbourhood of $\sqrt{p}$ in N and $\phi : S^{1/2} \to \Theta$ homeomorphism of $S^{1/2}$ onto an open subset Θ of $\mathbf{R}^m$. By considering the inverse map i of ϕ,

$$\begin{array}{rcl} i : \Theta & \longrightarrow & S^{1/2} \\ \theta & \longmapsto & \sqrt{p(\cdot,\theta)} \end{array}$$

we can express $S^{1/2}$ as

$$i(\Theta) = \{\sqrt{p(\cdot,\theta)}\,,\, \theta \in \Theta\} = S^{1/2}.$$

3.2 General manifolds

We shall denote by S the following family of probability densities :

$$S = \{p(\cdot,\theta)\,,\, \theta \in \Theta\},$$

where $\Theta \subseteq \mathbf{R}^m$ and we will work only with the single coordinate chart $(S^{1/2}, \phi)$ as it is done in [2]. From the fact that $(S^{1/2}, \phi)$ is a chart, it follows that

$$\{\frac{\partial i(\cdot,\theta)}{\partial \theta_1}, \cdots, \frac{\partial i(\cdot,\theta)}{\partial \theta_m}\}$$

is a set of linearly independent vectors in $L_2(\lambda)$. In such a context, let us see what the vectors of $L_{\sqrt{p(\cdot,\theta)}} S^{1/2}$ are. We can consider a curve in $S^{1/2}$ around $\sqrt{p(\cdot,\theta)}$ to be of the form $\gamma : h \mapsto \sqrt{p(\cdot,\theta(h))}$, where $h \mapsto \theta(h)$ is a curve in Θ around θ. Then, according to the chain rule, we compute the following Fréchet derivative:

$$D\gamma(0) = D\sqrt{p(\cdot,\theta(h))}\Big|_{h=0} = \sum_{k=1}^{m} \frac{\partial \sqrt{p(\cdot,\theta)}}{\partial \theta_k}\, \dot{\theta}_k(0) = \sum_{k=1}^{m} \frac{1}{2\sqrt{p(\cdot,\theta)}} \frac{\partial p(\cdot,\theta)}{\partial \theta_k}\, \dot{\theta}_k(0)\,.$$

We obtain that a basis for the tangent vector space at $\sqrt{p(\cdot,\theta)}$ to the space $S^{1/2}$ of square roots of densities of S is given by :

$$L_{\sqrt{p(\cdot,\theta)}} S^{1/2} = \mathrm{span}\{\frac{1}{2\sqrt{p(\cdot,\theta)}} \frac{\partial p(\cdot,\theta)}{\partial \theta_1}, \cdots, \frac{1}{2\sqrt{p(\cdot,\theta)}} \frac{\partial p(\cdot,\theta)}{\partial \theta_m}\}\,. \tag{3.1}$$

As i is the inverse of a chart, these vectors are actually linearly independent, and they indeed form a basis of the tangent vector space. One has to be careful, because if this were not true, the dimension of the above spanned space could drop. As an example, consider the curved exponential family

$$S = \{p(x,\theta) = \exp[-\theta_1^3 x - (\theta_2^2+1)x^2 - \psi(\theta)], \theta \in \Theta \subset \mathbf{R}^2\}$$

where ψ is the normalizing constant. It is immediate to check that at $(\theta_1, \theta_2) = (0,0)$ — assuming this point is in Θ — the linear space defined in (3.1) above reduces to a *one* dimensional subspace of L_2. This happens because $(S^{1/2}, \phi)$ is not a chart for the manifold N : it describes a different differential structure. The inner product of any two basis elements is defined, according to the L_2 inner product

$$\begin{aligned} &\langle \frac{1}{2\sqrt{p(\cdot,\theta)}} \frac{\partial p(\cdot,\theta)}{\partial \theta_i}, \frac{1}{2\sqrt{p(\cdot,\theta)}} \frac{\partial p(\cdot,\theta)}{\partial \theta_j} \rangle \\ &= \tfrac{1}{4} \int \frac{1}{p(x,\theta)} \frac{\partial p(x,\theta)}{\partial \theta_i} \frac{\partial p(x,\theta)}{\partial \theta_j} \, d\lambda(x) = \tfrac{1}{4}\, g_{ij}(\theta) \ . \end{aligned} \tag{3.2}$$

This is, up to the numeric factor $\frac{1}{4}$, the Fisher information metric, see [2], [49], and [1]. The matrix $g(\theta) = (g_{ij}(\theta))$ is called the Fisher information matrix.

Next, we introduce the orthogonal projection between any linear subspace V of $L_2(\lambda)$ containing the finite dimensional tangent vector space (3.1) and the tangent vector space (3.1) itself. Let us remember that our basis is not orthogonal, so that we have to project according to the following formula:

$$\begin{aligned} \Pi \ : V &\longrightarrow \operatorname{span}\{w_1, \cdots, w_m\} \\ v &\longmapsto \sum_{i=1}^m [\sum_{j=1}^m W^{ij} \ \langle v, w_j \rangle] \ w_i \end{aligned}$$

where $\{w_1, \cdots, w_m\}$ are m linearly independent vectors, $W := (\langle w_i, w_j \rangle)$ is the matrix formed by all the possible inner products of such linearly independent vectors, and (W^{ij}) is the inverse of the matrix W. In our context $\{w_1, \cdots, w_m\}$ are the vectors in (3.1), and of course W is, up to the numeric factor $\frac{1}{4}$, the Fisher information matrix given by (3.2) or (3.4). Then we obtain the following projection formula, where $(g^{ij}(\theta))$ is the inverse of the Fisher information matrix $(g_{ij}(\theta))$:

$$\Pi_\theta \ : L_2(\lambda) \supseteq V \longrightarrow \mathrm{span}\{\frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_1},\cdots,\frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_m}\}$$

$$\Pi_\theta[v] = \sum_{i=1}^{m}[\sum_{j=1}^{m} 4g^{ij}(\theta)\ \langle v, \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_j}\rangle]\ \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_i}\ . \tag{3.3}$$

Let us go back to the definition of tangent vectors for our statistical manifold. Amari [2] uses a different representation of tangent vectors to S at p. Without exploring all the assumptions needed, let us say that Amari defines an isomorphism between the actual tangent space and the vector space

$$\mathrm{span}\{\frac{\partial \log p(\cdot,\theta)}{\partial\theta_1},\cdots,\frac{\partial \log p(\cdot,\theta)}{\partial\theta_m}\}\ .$$

On this representation of the tangent space, Amari defines a Riemannian metric given by

$$E_{p(\cdot,\theta)}\{\frac{\partial \log p(\cdot,\theta)}{\partial\theta_i}\ \frac{\partial \log p(\cdot,\theta)}{\partial\theta_j}\}\ ,$$

where $E_p\{\cdot\}$ denotes the expectation w.r.t. the probability density p. This is again the Fisher information metric, and indeed this is the most frequent definition of Fisher metric. In fact, it is easy to check that

$$E_{p(\cdot,\theta)}\{\frac{\partial \log p(\cdot,\theta)}{\partial\theta_i}\ \frac{\partial \log p(\cdot,\theta)}{\partial\theta_j}\} = \int \frac{\partial \log p(x,\theta)}{\partial\theta_i}\ \frac{\partial \log p(x,\theta)}{\partial\theta_j}\ p(x,\theta)\,d\lambda(x) \tag{3.4}$$

$$= \int \frac{1}{p(x,\theta)}\ \frac{\partial p(x,\theta)}{\partial\theta_i}\ \frac{\partial p(x,\theta)}{\partial\theta_j}\ d\lambda(x) = g_{ij}(\theta)\ .$$

From the above relation and from (3.2) it is clear that, up to the numeric factor $\frac{1}{4}$, the Fisher information metric and the Hellinger metric coincide on the two representations of the tangent space to S at $p(\cdot,\theta)$.

There is another way of measuring how close two densities of S are. Consider the Kullback–Leibler information between two densities p and q of $H(\lambda)$:

$$K(p,q) := \int \log\frac{p(x)}{q(x)}\ p(x)\,d\lambda(x) = E_p\{\log\frac{p}{q}\}\ .$$

This is not a metric, since it is not symmetric and it does not satisfy the triangular inequality. When applied to a finite dimensional manifold such as S, both the Kullback–Leibler information and the Hellinger distance are particular cases of α–divergence, see [2] for the details. One can show that the Fisher metric and the Kullback–Leibler information coincide infinitesimally. Indeed, consider the two densities $p(\cdot,\theta)$ and $p(\cdot,\theta+d\theta)$ of S. By expanding in Taylor series, we obtain

$$\begin{aligned} K(p(\cdot,\theta),p(\cdot,\theta+d\theta)) &= -\sum_{i=1}^{m} E_{p(\cdot,\theta)}\{\frac{\partial \log p(\cdot,\theta)}{\partial\theta_i}\}\,d\theta_i \\ &\qquad -\sum_{i,j=1}^{m} E_{p(\cdot,\theta)}\{\frac{\partial^2 \log p(\cdot,\theta)}{\partial\theta_i\partial\theta_j}\}\,d\theta_i\,d\theta_j + O(|d\theta|^3) \\ &= \sum_{i,j=1}^{m} g_{ij}(\theta)\,d\theta_i\,d\theta_j + O(|d\theta|^3)\ . \end{aligned}$$

The interested reader is referred to [1].

3.3 Manifolds associated with exponential families

We conclude this section with some well known results about exponential families, which will be used in the following sections. More results on exponential families could be found in the books of Amari [2] and Barndorff–Nielsen [3]. We shall use the following equivalent notations for partial differentiation :

$$\frac{\partial^k}{\partial\theta_{i_1}\cdots\partial\theta_{i_k}} = \partial^k_{i_1,\cdots,i_k}\ .$$

Definition 3.3.1 *Let* $\{c_1,\cdots,c_m\}$ *be scalar functions* $c_i\ :\ \mathbf{R}^n \to \mathbf{R}$*,* $i=1,2,\cdots,m$ *such that* $\{1,c_1,\cdots,c_m\}$ *are* linearly independent*, and assume that the convex set*

$$\Theta_0 := \{\theta\in\mathbf{R}^m\ :\ \psi(\theta) = \log\int \exp[\theta^T c(x)]\,d\lambda(x) < \infty\}\ ,$$

has non–empty interior*. Then*

$$S = \{p(\cdot,\theta)\,,\,\theta\in\Theta\}, \qquad p(x,\theta) := \exp[\theta^T c(x) - \psi(\theta)]\ ,$$

where $\Theta\subseteq\Theta_0$ *is open, is called an exponential family of probability densities.*

Remark 3.3.2 *Given linearly independent scalar functions* $\{c_1,\cdots,c_m\}$ *defined on* $\mathbf{R}^n$*, it may happen that the densities* $\exp[\theta^T c(x)]$ *are not integrable. However, it is always possible to extend the family so as to deal with integrable densities only. Indeed, assume that there exist* $K>0$ *and* $r\geq 0$ *such that*

$$|c(x)| \leq K\,(1+|x|^r)\ ,$$

for all $x \in \mathbf{R}^n$*. Define* $d(x) := |x|^s$ *for all* $x \in \mathbf{R}^n$*, and some* $s > r$*. Then*

$$S' \quad := \quad \{p'(\cdot, \theta, \mu)\,,\, \theta \in \mathbf{R}^m, \mu > 0\},$$

$$p'(x, \theta, \mu) \quad := \quad \exp[\theta^T c(x) - \mu\, d(x) - \psi'(\theta, \mu)]\ ,$$

is an exponential family of densities, with a non–empty open parameter set.

Lemma 3.3.3 *Let*

$$S = \{p(\cdot, \theta)\,,\, \theta \in \Theta\}, \qquad p(x, \theta) := \exp[\theta^T c(x) - \psi(\theta)]\ ,$$

where $\Theta \subset \mathbf{R}^m$ *is open, be an exponential family of probability densities. Then the function* ψ *is infinitely differentiable in* Θ

$$E_{p(\cdot,\theta)}\{c_i\} = \partial_i \psi(\theta) =: \eta_i(\theta)\ ,$$

$$E_{p(\cdot,\theta)}\{c_i c_j\} = \partial^2_{ij}\psi(\theta) + \partial_i\psi(\theta)\,\partial_j\psi(\theta)\ ,$$

and more generally

$$E_{p(\cdot,\theta)}\{c_{i_1}\cdots c_{i_k}\} = \exp[-\psi(\theta)]\ \frac{\partial^k \exp[\psi(\theta)]}{\partial\theta_{i_1}\cdots\partial\theta_{i_k}}\ .$$

The Fisher information matrix satisfies

$$g_{ij}(\theta) = \partial^2_{ij}\psi(\theta) = \partial_i \eta_j(\theta)\ .$$

In the particular case where $n = 1$ *and*

$$c_i(x) = x^i\ , \qquad i = 1, \cdots, m$$

the following recursion formula holds, with $\eta_0(\theta) := 1$ *: for any nonnegative integer* i

$$\eta_{m+i}(\theta) := E_{p(\cdot,\theta)}\{x^{m+i}\} \tag{3.5}$$

$$= -\frac{1}{m\theta_m}\left[\begin{array}{ccccc} (i+1) & \theta_1 & 2\theta_2 & \cdots & (m-1)\theta_{m-1}\end{array}\right]\left[\begin{array}{c}\eta_i(\theta)\\ \eta_{i+1}(\theta)\\ \eta_{i+2}(\theta)\\ \vdots\\ \eta_{i+m-1}(\theta)\end{array}\right]\ .$$

Moreover, the entries of the Fisher information matrix satisfy

$$g_{ij}(\theta) = \eta_{i+j}(\theta) - \eta_i(\theta)\,\eta_j(\theta)\ . \tag{3.6}$$

PROOF : All these results may be found or immediately derived from Amari [2] (Chapter 4) or Barndorff-Nielsen [3] (Theorem 8.1). We only notice that some of the above properties follow easily by differentiating the identity

$$\int \exp[\theta^T c(x) - \psi(\theta)]\, dx = 1$$

w.r.t. the components $(\theta_1, \cdots, \theta_m)$ of θ. The particular recursion formula (3.5) is obtained via the following integration by parts:

$$\eta_i(\theta) = \int_{-\infty}^{+\infty} x^i \; p(x,\theta)\, dx$$

$$= [\frac{x^{i+1}}{i+1}\, p(x,\theta)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{x^{i+1}}{i+1}\, [\theta_1 + 2\theta_2\, x + \cdots + m\theta_m\, x^{m-1}]\; p(x,\theta)\, dx$$

$$= 0 - \frac{1}{i+1}\, E_{p(\cdot,\theta)}\{\theta_1\, x^{i+1} + 2\theta_2\, x^{i+2} + \cdots + m\theta_m\, x^{i+m}\}\ ,$$

from which the formula follows easily, remembering that $\eta_i(\theta) = E_{p(\cdot,\theta)}\{x^i\}$. □

Remark 3.3.4 *The quantities*

$$(\eta_1, \cdots, \eta_m) \in \mathcal{E} = \eta(\Theta) \subset \mathbf{R}^m$$

form a coordinate system for the given exponential family. The two coordinate systems θ (canonical parameters) and η (expectation parameters) are related by diffeomorphism, and according to the above results the Jacobian matrix of the transformation $\eta = \eta(\theta)$ is the Fisher information matrix. We shall use the notation $p_E(\cdot, \eta(\theta)) = p(\cdot,\theta)$ to express exponential densities of S as functions of the expectation parameters.

The canonical parameters and the expectation parameters are biorthogonal *w.r.t. the Fisher information metric : at $\sqrt{p(\cdot,\theta)} = \sqrt{p_E(\cdot,\eta)}$*

$$\langle \frac{\partial}{\partial\theta_i}\sqrt{p(\cdot,\theta)}, \frac{\partial}{\partial\eta_j}\sqrt{p_E(\cdot,\eta)}\rangle = \tfrac{1}{4}\,\delta_{ij}, \qquad i,j = 1,2,\cdots,m.$$

Chapter 4

The Projection Filter

Did we make a difference?
Captain James Kirk, Star Trek

4.1 Introduction

In this chapter we introduce the projection filter. We start by a section considering the projection in Fisher metric of the density evolution of a diffusion process onto a finite–dimensional manifold of densities. The projected density evolution is obtained via the projected Fokker–Planck equation. In that section we shall give more importance to intuition than to rigor. A rigorous setup for the projection of the Fokker–Planck equation will be developed in Chapter 7. Fully rigorous treatment of the more complicated projection of the filtering equation is given in the following sections. The filtering problem was introduced in Chapter 2. There we saw that from a probabilistic point of view, the filtering problem consists of the calculation of the complete conditional density of an unknown signal (state) given the observations up to the current instant. Such conditional density results in an infinite–dimensional filter in the general nonlinear case. Under some regularity conditions, the conditional density exists and is the solution of the Kushner–Stratonovich equation, a stochastic partial differential equation. In order to avoid infinite dimensionality, some approximation schemes have been proposed, yielding finite dimensional filters for the unobserved state. A well–known approximation method is the extended Kalman filter (EKF). The EKF is based upon linearization of the state equation around the current estimate, and application of the Kalman filter to the resulting linearized state equation. This procedure finds its justification in heuristic considerations, and not much is known about its performance, except in the case of small observation noise, see Picard [51], [52] and [53].

Another approximation method in the nonlinear case is the assumed–density filter (ADF), obtained from the selection of a few moment equations, which are closed under the assumption that the density is of a certain form, e.g. Gaussian, etc. We present a detailed definition of the assumed–density filters in Chapter 5. In the present chapter we introduce the projection filter (PF), which is a finite–dimensional nonlinear filter based on the differential–geometric approach to statistics. We particularize the PF to exponential families in the framework of SDE's on manifolds. The PF is obtained by orthogonally projecting the right–hand side of the Kushner–Stratonovich equation onto the tangent space of a finite–dimensional manifold of probability densities, according to the Fisher metric and its extension to infinite–dimensional space of square roots of densities, known as the Hellinger distance. We shall also present some formulae concerning auxiliary quantities, such as the projection residual (PR), the purpose of which is to provide a local measure of the quality of the filter behaviour. We develop explicit formulae for the particular example of the cubic sensor. The filters are derived by using the geometric approach, but in principle the reader can rederive them by using the assumed–density idea without using any Riemannian geometry, as we shall see in Chapter 5. Finally, we present some numerical simulations and comparisons for the cubic sensor, between the projection filter and the numerical solution of the nonlinear–filtering equation. Part of the material presented here has already appeared in [13] and [14].

4.2 Projection of the density evolution of a diffusion process

On the complete probability space $(\Omega, \mathcal{F}, P)$ let us consider a stochastic process $\{X_t, t \geq 0\}$ of diffusion type, adapted to a filtration $\{\mathcal{F}_t, t \geq 0\}$. Let the dynamic equation describing X be of the following form

$$dX_t = f_t(X_t)dt + \sigma(X_t)dW_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion independent of the initial condition X_0.

The equation above is an Itô stochastic differential equation. In the following derivations, in order to simplify notation, we treat the scalar case. The following sections will deal with the vector case. Precise assumptions on the coefficients f, σ, and h will also be given in the next sections.

Under suitable assumptions the law of X_t is absolutely continuous w.r.t. the Lebesgue measure and its density, i.e. $p_t(x)dx := P[X_t \in dx]$, satisfies the

Fokker–Planck equation (FPE):

$$\frac{\partial p_t}{\partial t} = \mathcal{L}_t^* p_t,$$

where the backward diffusion operator $\mathcal{L}_t$ is defined by

$$\mathcal{L}_t = f_t \frac{\partial}{\partial x} + \tfrac{1}{2} a_t \frac{\partial^2}{\partial x^2}.$$

At this point we introduce the geometric structure which permits to project the FPE onto a finite–dimensional manifold of densities. Again, the technical details can be found in the following sections, where we explain how to project a more complicate equation (namely the Kushner–Stratonovich equation for nonlinear filtering).

Rewrite FPE for the square root of p_t:

$$\frac{\partial \sqrt{p_t}}{\partial t} = \frac{\mathcal{L}_t^* p_t}{2\sqrt{p_t}}.$$

Next, select a finite dimensional manifold of square roots of densities to approximate $\sqrt{p_t}$. Let the family be parametrized by $\theta \in \Theta \subseteq \mathbf{R}^m$, where Θ is open. Call such manifold $S^{1/2}$,

$$S^{1/2} = \{\sqrt{p(\cdot,\theta)}, \theta \in \Theta\}.$$

Consider a generic curve $t \mapsto \sqrt{p(\cdot,\theta_t)}$ on this manifold. Its tangent vector in θ_t is given according to the chain rule:

$$\frac{d}{dt}\sqrt{p(\cdot,\theta_t)} = \sum_{i=1}^{m} \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_i}\, \dot{\theta}_t^{\,i} \ , \tag{4.1}$$

from which we see that tangent vectors in θ_t to all curves lie in the linear (tangent) space

$$\mathrm{span}\{\frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_1}, \ldots, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_m}\}.$$

Define the following quantity

$$g(\theta)_{ij} := 4\langle \frac{\partial \sqrt{p(\cdot,\theta)}}{\partial \theta_i}, \frac{\partial \sqrt{p(\cdot,\theta)}}{\partial \theta_j} \rangle, \quad i,j = 1,\ldots,m,$$

where $\langle\cdot,\cdot\rangle$ is the inner product of L_2. Notice that the matrix g is symmetric. The factor 4 is there for historical reasons, just to relate this L_2 structure to the traditional Fisher information metric $g(\theta)$ (see Section 3.2).

Now define for all $\theta \in \Theta$ the orthogonal projection

$$\Pi_\theta : L_2 \longrightarrow \mathrm{span}\{\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_1},\ldots,\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_m}\}$$
$$\Pi_\theta[v] := \sum_{i=1}^m [\sum_{j=1}^m 4g^{ij}(\theta)\,\langle v, \frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_j}\rangle]\,\frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_i}\ .$$

At this point we project the FPE equation for $\sqrt{p_t}$ via this projection, and we obtain the following (m–dimensional) SDE on the manifold $S^{1/2}$:

$$\frac{\partial}{\partial t}\sqrt{p(\cdot,\theta_t)} = \Pi_{\theta_t}[\frac{\mathcal{L}_t^* p(\cdot,\theta_t)}{2\sqrt{p(\cdot,\theta_t)}}].$$

Writing the projection map explicitly and comparing with (4.1) yields the following SDE for the parameters:

$$\frac{d}{dt}\theta_t = g^{-1}(\theta_t)\int \frac{\mathcal{L}_t^*\, p(x,\theta_t)}{p(x,\theta_t)}\,\frac{\partial p(x,\theta_t)}{\partial\theta}\,dx,$$

where integrals of vector functions are meant to be applied to their components. The equation above holds, under suitable assumptions, for any parametrized family of densities (not necessarily exponential). We will prove existence of the solution of such equation for exponential families. The point of this first section is to give to the reader a readable path of the basic steps involved in this 'projection idea'. The subsequent sections will formalize and generalize, in many ways, the results obtained so far.

4.3 General definition of the projection filter

In the present section we shall introduce the general definition of the projection filter. We begin by noticing that the stochastic calculus to be used in this derivation is the McShane–Fisk–Stratonovich (MFS) calculus. As remarked in Section 2.2, this is a standard choice for stochastic calculus on manifolds, as one can see for example in Elworthy [22], and is due to the fact that one does not know how to interpret second order terms arising in the Itô calculus in terms of manifold structures. For an account on McShane's integral, see the book [48].

We shall assume that the finite dimensional family $S^{1/2}$ we are working with has a manifold structure and a well defined Fisher information metric at all points $\theta \in \Theta$, according to the presentation given in Section 3.2. In order to project the Kushner–Stratonovich equation for $\sqrt{p_t}$ given in Section 2.2 onto the m–dimensional manifold $S^{1/2}$ we require the following assumption to be satisfied :

(DG) For all $\theta \in \Theta$ and all $t \geq 0$

$$E_{p(\cdot,\theta)}\{|\frac{\mathcal{L}_t^* p(\cdot,\theta)}{p(\cdot,\theta)}|^2\} < \infty \qquad \text{and} \qquad E_{p(\cdot,\theta)}\{|h_t|^4\} < \infty \ .$$

These assumptions will be explored in detail for exponential families in Section 4.4, and explicit sufficient conditions under which they hold will be given. These conditions ensure that for all $\theta \in \Theta$ and all $t \geq 0$, the vectors $\mathcal{P}_t(\sqrt{p(\cdot,\theta)})$ and $\mathcal{Q}_t^k(\sqrt{p(\cdot,\theta)})$ (see definitions of Section 2.2) for $k = 0, 1, \cdots, d$ are vectors in $L_2(\lambda)$, so that indeed the projection can take place according to the $L_2(\lambda)$ structure described in Sections 3.1 and 3.2.

The projection filter for the family $S = \{p(\cdot,\theta)\,,\, \theta \in \Theta\}$ is defined as the solution of the following stochastic differential equation on the manifold $S^{1/2}$:

$$d\sqrt{p(\cdot,\theta_t)} = \Pi_{\theta_t} \circ \mathcal{P}_t(\sqrt{p(\cdot,\theta_t)})\, dt - \Pi_{\theta_t} \circ \mathcal{Q}_t^0(\sqrt{p(\cdot,\theta_t)})\, dt$$

$$+ \sum_{k=1}^{d} \Pi_{\theta_t} \circ \mathcal{Q}_t^k(\sqrt{p(\cdot,\theta_t)}) \circ dY_t^k \ , \tag{4.2}$$

where for all $\theta \in \Theta$, the projection map Π_θ is defined in (3.3).

Remark 4.3.1 *Notice that although at first sight (4.2) may look like a stochastic partial differential equation (PDE), it is just a finite dimensional SDE which can be equivalently written using different coordinates as an equation in $\Theta \subset \mathbf{R}^m$ for the parameter θ_t. The explicit form of this SDE is given in the following theorem.*

Theorem 4.3.2 *Assume that, in addition to (A), (B) and (C), the coefficients f_t, a_t and h_t of the system (2.1), and the family S satisfy (DG), i.e.*

$$E_{p(\cdot,\theta)}\{|\frac{\mathcal{L}_t^* p(\cdot,\theta)}{p(\cdot,\theta)}|^2\} < \infty \qquad \textit{and} \qquad E_{p(\cdot,\theta)}\{|h_t|^4\} < \infty \ ,$$

holds for all $\theta \in \Theta$, and all $t \geq 0$. Assume $p_0(\cdot) = p(\cdot,\theta_0) \in S$.

Then, for all $\theta \in \Theta$ and all $t \geq 0$, the vectors $\mathcal{P}_t(\sqrt{p(\cdot,\theta)})$ and $\mathcal{Q}_t^k(\sqrt{p(\cdot,\theta)})$ for $k = 0, 1, \cdots, d$ are vectors in $L_2(\lambda)$.

For all $\theta \in \Theta$ and all $t \geq 0$, the nonlinear operators $\Pi_\theta \circ \mathcal{P}_t$ and $\Pi_\theta \circ \mathcal{Q}_t^k$ for $k = 0, 1, \cdots, d$ are vector fields on the manifold $S^{1/2}$, where the projection map Π_θ is defined in (3.3).

The projection filter density $p(\cdot,\theta_t)$ is described by equation (4.2), and the projection filter parameters satisfy the following stochastic differential equation :

$$
\begin{aligned}
d\theta_t^i \;=\;& [\sum_{j=1}^m g^{ij}(\theta_t) \int \frac{\mathcal{L}_t^* \, p(x,\theta_t)}{p(x,\theta_t)} \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x)] \, dt \\
& - [\sum_{j=1}^m g^{ij}(\theta_t) \int \tfrac{1}{2} \, |h_t(x)|^2 \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x)] \, dt \qquad (4.3) \\
& + \sum_{k=1}^d [\sum_{j=1}^m g^{ij}(\theta_t) \int h_t^k(x) \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x)] \circ dY_t^k \;, \quad \theta_0^i \; .
\end{aligned}
$$

Under the assumptions on the coefficients, this equation has a unique solution up to the a.s. positive exit time $\tau := \inf\{t \geq 0 \, : \, \theta_t \notin \Theta\}$.

Proof : Let us compute the projections of the operators in the right-hand side of the Kushner-Stratonovich equation. Under assumption (DG) such projections always exist.

$$
\begin{aligned}
& \Pi_{\theta_t} \circ \mathcal{P}_t(\sqrt{p(\cdot,\theta_t)}) = \Pi_{\theta_t}[\, \frac{\mathcal{L}_t^* \, p(\cdot,\theta_t)}{2\sqrt{p(\cdot,\theta_t)}} \,] = \\
=& \sum_{i=1}^m [\sum_{j=1}^m 4g^{ij}(\theta_t) \, \langle \frac{\mathcal{L}_t^* \, p(\cdot,\theta_t)}{2\sqrt{p(\cdot,\theta_t)}}, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_j} \rangle \,] \, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_i} \\
=& \sum_{i=1}^m [\sum_{j=1}^m g^{ij}(\theta_t) \int \frac{\mathcal{L}_t^* \, p(x,\theta_t)}{p(x,\theta_t)} \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x)] \, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_i} \; .
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& \Pi_{\theta_t} \circ \mathcal{Q}_t^0(\sqrt{p(\cdot,\theta_t)}) = \Pi_{\theta_t}[\, \tfrac{1}{2} \, \sqrt{p(\cdot,\theta_t)} \; \beta_t^0(p(\cdot,\theta_t)) \,] = \\
=& \sum_{i=1}^m [\sum_{j=1}^m 4g^{ij}(\theta_t) \, \langle \tfrac{1}{2} \, \sqrt{p(\cdot,\theta_t)} \; \beta_t^0(p(\cdot,\theta_t)), \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_j} \rangle \,] \, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_i} \\
=& \sum_{i=1}^m [\sum_{j=1}^m g^{ij}(\theta_t) \int \tfrac{1}{2} \, [|h_t(x)|^2 - E_{p(\cdot,\theta_t)}\{|h_t|^2\}] \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x)] \, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_i} \\
=& \sum_{i=1}^m [\sum_{j=1}^m g^{ij}(\theta_t) \int \tfrac{1}{2} \, |h_t(x)|^2 \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x)] \, \frac{\partial \sqrt{p(\cdot,\theta_t)}}{\partial \theta_i} \; ,
\end{aligned}
$$

and

$$
\Pi_{\theta_t} \circ \mathcal{Q}_t^k(\sqrt{p(\cdot,\theta_t)}) = \Pi_{\theta_t} \, [\tfrac{1}{2} \, \sqrt{p(\cdot,\theta_t)} \; \beta_t^k(p(\cdot,\theta_t)) \,] =
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} [\sum_{j=1}^{m} 4g^{ij}(\theta_t)\, \langle \tfrac{1}{2}\, \sqrt{p(\cdot,\theta_t)}\ \beta_t^k(p(\cdot,\theta_t)), \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_j} \rangle]\ \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i} \\
&= \sum_{i=1}^{m} [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int [h_t^k(x) - E_{p(\cdot,\theta_t)}\{h_t^k\}]\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i} \\
&= \sum_{i=1}^{m} [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int h_t^k(x)\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i}\ ,
\end{aligned}
$$

for $k = 1, \cdots, d$. We have used the fact that the constant terms $E_{p(\cdot,\theta_t)}\{|h_t|^2\}$ and $E_{p(\cdot,\theta_t)}\{h_t^k\}$ give no contribution to the projection, since

$$
\int \frac{\partial p(x,\theta_t)}{\partial\theta_i}\, d\lambda(x) = 0\ .
$$

We conclude by rewriting equation (4.2) in the more detailed form

$$
\begin{aligned}
d\sqrt{p(\cdot,\theta_t)} &= \sum_{i=1}^{m} [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int \frac{\mathcal{L}_t^*\, p(x,\theta_t)}{p(x,\theta_t)}\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i}\ dt \\
&- \sum_{i=1}^{m} [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int \tfrac{1}{2}\, |h_t(x)|^2\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i}\, dt \\
&+ \sum_{i=1}^{m} \sum_{k=1}^{d} [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int h_t^k(x)\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i} \circ dY_t^k\ .
\end{aligned}
\tag{4.4}
$$

By expanding $\sqrt{p(\cdot,\theta_t)}$ according to the Stratonovich chain rule

$$
d\sqrt{p(\cdot,\theta_t)} = \sum_{i=1}^{m} \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i} \circ d\theta_t^i\ ,
$$

and comparing with (4.4) we obtain the following equation for the parameters θ_t describing our projected density in S :

$$
\begin{aligned}
d\theta_t^i &= [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int \frac{\mathcal{L}_t^*\, p(x,\theta_t)}{p(x,\theta_t)}\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ dt \\
&- [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int \tfrac{1}{2}\, |h_t(x)|^2\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)]\ dt \\
&+ \sum_{k=1}^{d} [\sum_{j=1}^{m} g^{ij}(\theta_t)\ \int h_t^k(x)\ \frac{\partial p(x,\theta_t)}{\partial\theta_j}\ d\lambda(x)] \circ dY_t^k\ ,\ \ \theta_0^i,
\end{aligned}
$$

which is our finite dimensional filter. Consider the term

$$\int \frac{\mathcal{L}_t^* \, p(x,\theta_t)}{p(x,\theta_t)} \, \frac{\partial p(x,\theta_t)}{\partial \theta_j} \, d\lambda(x) \ .$$

Usually, as in the case of the optimal filter, terms involving $\mathcal{L}_t^*$ cause infinite–dimensionality (due to differential operators in x involved in $\mathcal{L}_t^*$). Nonetheless, this problem is not affecting our approximated filter. Indeed, the integral above reduces to a function of θ_t, which is a finite dimensional parameter.

□

4.4 The exponential projection filter

In this section we present the rigorous definition of an exponential projection filter. We will show that if we choose $S^{1/2}$ as the square roots of a finite dimensional exponential family, then under some additional assumptions the operators $\mathcal{P}_t$ and $\mathcal{Q}_t^k$ for $k = 0, 1, \cdots, d$ introduced in (2.8) define at each point $\sqrt{p(\cdot,\theta)} \in S^{1/2}$ tangent vectors to a larger but *finite dimensional* (smoothly embedded) submanifold $\Sigma_{t,\theta}^{1/2}$ of $L_2(\lambda)$, whose elements are (square roots of) exponential densities of a larger (curved) exponential family. The manifolds $\Sigma_{t,\theta}^{1/2}$ may be viewed as enveloping manifolds for $S^{1/2}$. Within such a setup, the projection can take place within a *finite dimensional* tangent space, and infinite dimensionality is bypassed. The additional conditions we shall impose are necessary to ensure that the operator $\mathcal{P}_t$ takes values in $L_2(\lambda)$, so that we can actually project the coefficients in the right hand side of the Kushner–Stratonovich equation, according to formula (3.3).

Let us consider the following exponential family of probability densities

$$S := \{p(\cdot,\theta)\,,\, \theta \in \Theta\}, \qquad p(x,\theta) := \exp[\theta^T c(x) - \psi(\theta)] \ , \tag{4.5}$$

where $\Theta \subseteq \mathbf{R}^m$ is open. According to (2.5), we define

$$\begin{aligned}
\alpha_{t,\theta} &:= \alpha_t(p(\cdot,\theta)) = \frac{\mathcal{L}_t^* \, p(\cdot,\theta)}{p(\cdot,\theta)} \ , \\
\beta_{t,\theta}^0 &:= \beta_t^0(p(\cdot,\theta)) = \tfrac{1}{2}\,[|h_t|^2 - E_{p(\cdot,\theta)}\{|h_t|^2\}] \\
\beta_{t,\theta}^k &:= \beta_t^k(p(\cdot,\theta)) = h_t^k - E_{p(\cdot,\theta)}\{h_t^k\} \ ,
\end{aligned}$$

for $k = 1, \cdots, d$. From the expression obtained in (2.6), it follows that

$$\alpha_{t,\theta} = -\sum_{i=1}^{n} [\, f_t^i \frac{\partial}{\partial x_i}(\theta^T c) + \frac{\partial f_t^i}{\partial x_i} \,]$$

$$+ \tfrac{1}{2} \sum_{i,j=1}^{n} [\, a_t^{ij} \frac{\partial^2}{\partial x_i \partial x_j}(\theta^T c) + a_t^{ij} \frac{\partial}{\partial x_i}(\theta^T c) \frac{\partial}{\partial x_j}(\theta^T c) \tag{4.6}$$

$$+2 \frac{\partial a_t^{ij}}{\partial x_j} \frac{\partial}{\partial x_i}(\theta^T c) + \frac{\partial^2 a_t^{ij}}{\partial x_i \partial x_j} \,] \;.$$

We make the following additional assumption on the coefficients f_t, a_t and h_t of the system (2.1), and on the coefficients c of the exponential family (4.5)

(D) For all $\theta \in \Theta$ and all $t \geq 0$

$$E_{p(\cdot,\theta)}\{|\alpha_{t,\theta}|^2\} < \infty \qquad \text{and} \qquad E_{p(\cdot,\theta)}\{|h_t|^4\} < \infty \;.$$

Under the assumption (D) we define below, for any $\theta_0 \in \Theta$ and $t_0 \geq 0$, a *curved* exponential family Σ_{t_0,θ_0}, containing S. For the definition of a curved exponential family, see [2, Section 4.2].

Proposition 4.4.1 *Let* $\{d_1, \cdots, d_s\}$, *with* $0 \leq s \leq d+2$, $d_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \cdots, s$ *be scalar functions depending on* t_0, θ_0, *such that* $\{1, c_1, \cdots, c_m, d_1, \cdots, d_s\}$ *is a basis of the linear space*

$$\operatorname{span}\{1, c_1, \cdots, c_m, \alpha_{t_0,\theta_0}, \tfrac{1}{2}\,|h_{t_0}|^2, h_{t_0}^1, \cdots, h_{t_0}^d\} \;.$$

Define

$$\Sigma_{t_0,\theta_0} := \{p_{t_0,\theta_0}(\cdot,\theta,\xi)\,,\; \theta \in \Theta\,,\; \xi \in \Xi\} \;, \tag{4.7}$$

with

$$p_{t_0,\theta_0}(x,\theta,\xi) := \exp[\theta^T c(x) + \xi^T d(x) - \tfrac{1}{4}\,|\xi|^4\,|d(x)|^4 - \psi_{t_0,\theta_0}(\theta,\xi)] \;, \tag{4.8}$$

and where $\Xi \subseteq \mathbf{R}^s$ *is an open set. If the assumption (D) holds, and if* $\Xi \subseteq \mathbf{R}^s$ *is a sufficiently small neighbourhood of the origin, then* $\Sigma_{t_0,\theta_0}^{1/2}$ *is a* $(m+s)$*–dimensional submanifold of* $L_2(\lambda)$.

Remark 4.4.2 *For any* $\theta \in \Theta$, $p(\cdot,\theta) = p_{t_0,\theta_0}(\cdot,\theta,0)$, *hence* $S \subset \Sigma_{t_0,\theta_0}$, *which makes* $\Sigma_{t_0,\theta_0}^{1/2}$ *an* enveloping manifold *of* $S^{1/2}$. *Here enveloping is meant in the sense that all curves on* $S^{1/2}$ *are particular curves on* Σ_{t_0,θ_0}, *and tangent vectors to curves in* $S^{1/2}$ *are particular tangent vectors to particular curves of* Σ_{t_0,θ_0}.

PROOF : For simplicity, we use in this proof the notations $p_0(\cdot,\theta,\xi) = p_{t_0,\theta_0}(\cdot,\theta,\xi)$, and $\psi_0(\theta,\xi) = \psi_{t_0,\theta_0}(\theta,\xi)$.

It follows from the Cauchy–Schwartz inequality, and the Young inequality : $u \leq \frac{3}{4} + \frac{1}{4}\, u^4, \quad u \in \mathbf{R}$, that

$$p_0(x,\theta,\xi) \leq \exp[\theta^T c(x) + \tfrac{3}{4} - \psi_0(\theta,\xi)] \; ,$$

hence $p_0(\cdot,\theta,\xi)$ is integrable for any $\theta \in \Theta$, and any $\xi \in \mathbf{R}^s$.

Define the following *expectation* parameters

$$\begin{aligned}
\bar{\eta}_i(\theta,\xi) \quad &:= \quad \frac{\partial}{\partial\theta_i}\psi_0(\theta,\xi) = E_{p_0(\cdot,\theta,\xi)}\{c_i\} \; , \qquad i = 1,\cdots,m \; , \\
\bar{\chi}_l(\theta,\xi) \quad &:= \quad \frac{\partial}{\partial\xi_l}\psi_0(\theta,\xi) = E_{p_0(\cdot,\theta,\xi)}\{d_l - \xi_l\,|\xi|^2\,|d|^4\} \; , \qquad l = 1,\cdots,s \; ,
\end{aligned}$$

and the tangent vectors

$$\begin{aligned}
\frac{\partial}{\partial\theta_i}\sqrt{p_0(\cdot,\theta,\xi)} \quad &= \quad \tfrac{1}{2}\,\sqrt{p_0(\cdot,\theta,\xi)}\,[c_i - \bar{\eta}_i(\theta,\xi)] \; , \qquad i = 1,\cdots,m \; , \\
\frac{\partial}{\partial\xi_l}\sqrt{p_0(\cdot,\theta,\xi)} \quad &= \quad \tfrac{1}{2}\,\sqrt{p_0(\cdot,\theta,\xi)}\,[d_l - \xi_l\,|\xi|^2\,|d|^4 - \bar{\chi}_l(\theta,\xi)] \qquad l = 1,\cdots,s \; ,
\end{aligned}$$

at point $\sqrt{p_0(\cdot,\theta,\xi)} \in \Sigma^{1/2}_{t_0,\theta_0}$. Under the assumption (D), it holds

$$\begin{aligned}
E_{p_0(\cdot,\theta,\xi)}\{|d|^2\} \quad &= \quad E_{p(\cdot,\theta)}\{|d|^2\ \exp[\xi^T d - \tfrac{1}{4}\,|\xi|^4\,|d|^4]\}\ \exp[\psi(\theta) - \psi_0(\theta,\xi)] \\
&\leq \quad E_{p(\cdot,\theta)}\{|d|^2\}\ \exp[\tfrac{3}{4} + \psi(\theta) - \psi_0(\theta,\xi)] < \infty \; ,
\end{aligned}$$

and similarly

$$\begin{aligned}
&|\xi|^6\ E_{p_0(\cdot,\theta,\xi)}\{|d|^8\} \\
&= E_{p(\cdot,\theta)}\{|\xi|^6\ |d|^8\ \exp[\xi^T d - \tfrac{1}{4}\,|\xi|^4\,|d|^4]\}\ \exp[\psi(\theta) - \psi_0(\theta,\xi)] \\
&\leq E_{p(\cdot,\theta)}\{|d|^2\}\ \max_{u\geq 0}\{u^6\ \exp[u - \tfrac{1}{4}\,u^4]\}\ \exp[\psi(\theta) - \psi_0(\theta,\xi)] < \infty \; ,
\end{aligned}$$

which proves that all the tangent vectors introduced above are in $L_2(\lambda)$, and hence the associated Fisher information matrix $\bar{g}(\theta,\xi)$ is well defined.

Finally, it is easy to prove that these tangent vectors are linearly independent, and hence the Fisher information matrix is invertible. Indeed, the following decomposition holds

$$|d(x)|^4 = \alpha + \beta^T c(x) + \gamma^T d(x) + e(x) \; ,$$

where the scalar function e either is zero, or is linearly independent of

$$\{1, c_1, \cdots, c_m, d_1, \cdots, d_s\},$$

and

$$\begin{aligned} 0 &= \rho + \lambda^T [c - \bar\eta(\theta,\xi)] + \mu^T [d - \xi\, |\xi|^2\, |d|^4 - \bar\chi(\theta,\xi)] \\ &= [\rho - \lambda^T \bar\eta(\theta,\xi) - \mu^T \bar\chi(\theta,\xi) - \mu^T \xi\, |\xi|^2\, \alpha] + [\lambda - \mu^T \xi\, |\xi|^2\, \beta]^T c \\ &\qquad + [\mu - \mu^T \xi\, |\xi|^2\, \gamma]^T d - \mu^T \xi\, |\xi|^2\, e\ , \end{aligned}$$

implies

$$\begin{aligned} \rho - \lambda^T \bar\eta(\theta,\xi) - \mu^T \bar\chi(\theta,\xi) - \mu^T \xi\, |\xi|^2\, \alpha &= 0 \\ \lambda - \mu^T \xi\, |\xi|^2\, \beta &= 0 \\ [I - \gamma\, \xi^T\, |\xi|^2]\, \mu = \mu - \mu^T \xi\, |\xi|^2\, \gamma &= 0\ . \end{aligned}$$

If ξ is sufficiently small, the matrix $[I - \gamma\, \xi^T\, |\xi|^2]$ is invertible, hence $\mu = 0$, from which we deduce $\lambda = 0$, and $\rho = 0$. □

It is easily checked that for all $\theta \in \Theta$

$$\operatorname{span}\{\tfrac12 \sqrt{p(\cdot,\theta)}\, \alpha_{t_0,\theta_0}, \tfrac12 \sqrt{p(\cdot,\theta)}\, \beta^k_{t_0,\theta_0}\ ,\ k = 0, 1, \cdots, d\} \subseteq L_{\sqrt{p(\cdot,\theta)}} \Sigma^{1/2}_{t_0,\theta_0}\ .$$

Let us consider the equation (2.7) in MFS form for $\{\sqrt{p_t}\ ,\ t \geq t_0\}$, starting at time t_0 from the initial condition $\sqrt{p_{t_0}} = \sqrt{p(\cdot,\theta_0)} \in S^{1/2}$ for some $\theta_0 \in \Theta$, i.e.

$$\begin{aligned} d\sqrt{p_t} &= \tfrac12 \sqrt{p_t}\, \alpha_t(p_t)\, dt - \tfrac12 \sqrt{p_t}\, \beta^0_t(p_t)\, dt + \tfrac12 \sum_{k=1}^d \sqrt{p_t}\, \beta^k_t(p_t) \circ dY^k_t \\ &= \mathcal{P}_t(\sqrt{p_t})\, dt - \mathcal{Q}^0_t(\sqrt{p_t})\, dt + \sum_{k=1}^d \mathcal{Q}^k_t(\sqrt{p_t}) \circ dY^k_t\ , \qquad t \geq t_0\ . \end{aligned} \tag{4.9}$$

It is immediate to check that

$$\mathcal{P}_{t_0}(\sqrt{p_{t_0}}) = \tfrac12 \sqrt{p_{t_0}}\, \alpha_{t_0}(p_{t_0}) = \tfrac12 \sqrt{p(\cdot,\theta_0)}\, \alpha_{t_0,\theta_0} \in L_{\sqrt{p(\cdot,\theta_0)}} \Sigma^{1/2}_{t_0,\theta_0}\ ,$$

and

$$\mathcal{Q}^k_{t_0}(\sqrt{p_{t_0}}) = \tfrac12 \sqrt{p_{t_0}}\, \beta^k_{t_0}(p_{t_0}) = \tfrac12 \sqrt{p(\cdot,\theta_0)}\, \beta^k_{t_0,\theta_0} \in L_{\sqrt{p(\cdot,\theta_0)}} \Sigma^{1/2}_{t_0,\theta_0}\ ,$$

for $k = 0, 1, \cdots, d$. Then we can project at any time instant t_0 from the finite dimensional tangent vector space $L_{\sqrt{p(\cdot,\theta_0)}} \Sigma^{1/2}_{t_0,\theta_0}$ onto the finite dimensional tangent vector space $L_{\sqrt{p(\cdot,\theta_0)}} S^{1/2}$ since the Fisher metric in the enveloping manifold is well defined under the assumption (D).

Let $\langle \cdot, \cdot \rangle$ be the Fisher information metric on the enveloping manifold at the current point $p(\cdot,\theta_0) = p_{t_0,\theta_0}(\cdot,\theta,0)$. Consider the orthogonal projection

$$\begin{aligned} \Pi_{t_0,\theta_0} \quad &: \quad L_{\sqrt{p(\cdot,\theta_0)}} \Sigma^{1/2}_{t_0,\theta_0} \longrightarrow L_{\sqrt{p(\cdot,\theta_0)}} S^{1/2} \\ v \quad &\longmapsto \sum_{i=1}^m [\sum_{j=1}^m 4 g^{ij}(\theta_0) \, \langle v, \frac{1}{2\sqrt{p(\cdot,\theta_0)}} \frac{\partial p(\cdot,\theta_0)}{\partial \theta_j} \rangle] \, \frac{1}{2\sqrt{p(\cdot,\theta_0)}} \frac{\partial p(\cdot,\theta_0)}{\partial \theta_i} \; . \end{aligned} \tag{4.10}$$

The exponential projection filter for the exponential family S is defined as the solution of the following stochastic differential equation on the manifold $S^{1/2}$:

$$\begin{aligned} d\sqrt{p(\cdot,\theta_t)} \quad &= \quad \Pi_{t,\theta_t} \circ \mathcal{P}_t(\sqrt{p(\cdot,\theta_t)}) \, dt - \Pi_{t,\theta_t} \circ \mathcal{Q}^0_t(\sqrt{p(\cdot,\theta_t)}) \, dt \\ &\quad + \sum_{k=1}^d \Pi_{t,\theta_t} \circ \mathcal{Q}^k_t(\sqrt{p(\cdot,\theta_t)}) \circ dY^k_t \; . \end{aligned} \tag{4.11}$$

Of course the operators

$$\begin{aligned} S^{1/2} \quad &\longrightarrow \quad L\, S^{1/2} \\ \sqrt{p(\cdot,\theta)} \quad &\longmapsto \quad \Pi_{t,\theta} \circ \mathcal{P}_t(\sqrt{p(\cdot,\theta)}) \in L_{\sqrt{p(\cdot,\theta)}} S^{1/2} \; , \end{aligned}$$

and

$$\begin{aligned} S^{1/2} \quad &\longrightarrow \quad L\, S^{1/2} \\ \sqrt{p(\cdot,\theta)} \quad &\longmapsto \quad \Pi_{t,\theta} \circ \mathcal{Q}^k_t(\sqrt{p(\cdot,\theta)}) \in L_{\sqrt{p(\cdot,\theta)}} S^{1/2} \; , \end{aligned}$$

for $k = 0, 1, \cdots, d$, are vector fields on the manifold $S^{1/2}$.

We can now state the main result of this section.

Theorem 4.4.3 *Assume that, in addition to (A), (B) and (C), the coefficients f_t, a_t and h_t of the system (2.1), and the coefficients c of the exponential family (4.5) satisfy (D), i.e.*

$$E_{p(\cdot,\theta)}\{|\alpha_{t,\theta}|^2\} < \infty \qquad \textit{and} \qquad E_{p(\cdot,\theta)}\{|h_t|^4\} < \infty \; ,$$

holds for all $\theta \in \Theta$, and all $t \geq 0$, where the expression of $\alpha_{t,\theta}$ is given in (4.6).

Then, for all $\theta \in \Theta$ and all $t \geq 0$, the vectors $\mathcal{P}_t(\sqrt{p(\cdot,\theta)})$ and $\mathcal{Q}_t^k(\sqrt{p(\cdot,\theta)})$ for $k = 0, 1, \cdots, d$ are tangent vectors of a $(m+s)$–dimensional, with $0 \leq s \leq d+2$, time varying submanifold $\Sigma_{t,\theta}^{1/2}$ of $L_2(\lambda)$, where $\Sigma_{t,\theta}$ is the curved exponential family defined in (4.7) and (4.8).

Let $\Pi_{t,\theta}$ denote the projection map defined in (4.10). The nonlinear operators $\Pi_{t,\theta} \circ \mathcal{P}_t$ and $\Pi_{t,\theta} \circ \mathcal{Q}_k$ for $k = 0, 1, \cdots, d$ are vector fields on the original exponential manifold $S^{1/2}$.

The projection filter density $p(\cdot, \theta_t)$ is described by equation (4.11), and the projection filter parameters satisfy the following stochastic differential equation :

$$\begin{aligned} g(\theta_t) \circ d\theta_t &= E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c\}\, dt - E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\, |h_t|^2\, [c - \eta(\theta_t)]\}\, dt \\ &\quad + \sum_{k=1}^{d} E_{p(\cdot,\theta_t)}\{h_t^k\, [c - \eta(\theta_t)]\} \circ dY_t^k \ , \quad \theta_0 \ . \end{aligned} \tag{4.12}$$

Under the assumptions on the coefficients, this equation has a unique solution up to the a.s. positive exit time $\tau := \inf\{t \geq 0 \,:\, \theta_t \notin \Theta\}$.

Remark 4.4.4 *The question of whether the exit time τ is a.s. finite or infinite will be addressed in future research work.*

Remark 4.4.5 *The weaker conditions*

$$E_{p(\cdot,\theta)}\{|\mathcal{L}_t\, c|\} < \infty \qquad \text{and} \qquad E_{p(\cdot,\theta)}\{|h_t|^2\} < \infty \ ,$$

for all $\theta \in \Theta$, and all $t \geq 0$, are sufficient for proving existence and uniqueness of a solution to equation (4.12) up to the exit time τ. The question of whether the interpretation as a projected equation still holds under these weaker conditions will require further investigation.

Remark 4.4.6 *Notice that although (4.11) at a first sight may look like a stochastic PDE, it is just a finite dimensional SDE for the parameter θ_t expressed in different coordinates. The explicit form of this SDE is given by (4.12).*

PROOF : Consider equation (4.3) derived in Section 4.3. For the special case of the exponential family introduced above in (4.5), we obtain

$$\begin{aligned} d\theta_t^i &= [\sum_{j=1}^m g^{ij}(\theta_t) \int \mathcal{L}_t\, c_j(x)\; p(x,\theta_t)\; d\lambda(x)]\; dt \\ &\quad - [\sum_{j=1}^m g^{ij}(\theta_t) \int \tfrac{1}{2}\, |h_t(x)|^2\; [c_j(x) - \eta_j(\theta_t)]\; p(x,\theta_t)\; d\lambda(x)]\; dt \\ &\quad + \sum_{k=1}^d [\sum_{j=1}^m g^{ij}(\theta_t) \int h_t^k(x)\; [c_j(x) - \eta_j(\theta_t)]\; p(x,\theta_t)\; d\lambda(x)] \circ dY_t^k \; . \end{aligned} \tag{4.13}$$

We have used the following duality relation

$$\begin{aligned} \int \frac{\mathcal{L}_t^*\, p(x,\theta_t)}{p(x,\theta_t)}\; \frac{\partial p(x,\theta_t)}{\partial \theta_j}\; d\lambda(x) &= \int \mathcal{L}_t^*\, p(x,\theta_t)\; [c_j(x) - \eta_j(\theta_t)]\; d\lambda(x) \\ &= \int \mathcal{L}_t\, c_j(x)\; p(x,\theta_t)\; d\lambda(x) \; . \end{aligned}$$

Another way of writing equation (4.13) is

$$\begin{aligned} d\theta_t^i &= [\sum_{j=1}^m g^{ij}(\theta_t)\; E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c_j\}]\; dt \\ &\quad -[\textstyle\sum_{j=1}^m g^{ij}(\theta_t)\; E_{p(\cdot,\theta_t)}\{\frac{1}{2}\, |h_t|^2\; [c_j - \eta_j(\theta_t)]\}]\; dt \\ &\quad + \sum_{k=1}^d [\sum_{j=1}^m g^{ij}(\theta_t)\; E_{p(\cdot,\theta_t)}\{h_t^k\; [c_j - \eta_j(\theta_t)]\}] \circ dY_t^k \; . \end{aligned} \tag{4.14}$$

In vector form, the above equation reads

$$\begin{aligned} d\theta_t &= [g(\theta_t)]^{-1}\, E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c\}\, dt - [g(\theta_t)]^{-1}\, E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\, |h_t|^2\, [c - \eta(\theta_t)]\}\, dt \\ &\quad + [g(\theta_t)]^{-1} \sum_{k=1}^d\; E_{p(\cdot,\theta_t)}\{h_t^k\, [c - \eta(\theta_t)]\} \circ dY_t^k \; , \end{aligned}$$

or equivalently

$$\begin{aligned} g(\theta_t) \circ d\theta_t &= E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c\}\, dt - E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\, |h_t|^2\, [c - \eta(\theta_t)]\}\, dt \\ &\quad + \sum_{k=1}^d E_{p(\cdot,\theta_t)}\{h_t^k\, [c - \eta(\theta_t)]\} \circ dY_t^k \; . \end{aligned}$$

Under the assumptions, the mappings $\theta \mapsto E_{p(\cdot,\theta)}\{\mathcal{L}_t\, c\}$, and $\theta \mapsto E_{p(\cdot,\theta)}\{\frac{1}{2}\, |h_t|^2\}$ are locally Lipschitz continuous. Therefore, there exists a unique solution to equation (4.12) up to the a.s. positive exit time τ, see [37]. □

Remark 4.4.7 *The initial condition θ_0 for equation (4.12) is defined as follows : If $p_0 \in S$, then $p_0 = p(\cdot, \theta_0)$ for some unique $\theta_0 \in \Theta$, which is used as the initial condition. Otherwise, we project p_0 on S, by minimizing the Kullback–Leibler information*

$$K(p_0, p(\cdot, \theta)) := \int \log \frac{p_0(x)}{p(x,\theta)}\, p_0(x)\, d\lambda(x)\ ,$$

w.r.t. $\theta \in \Theta$. After straightforward calculations, and making use of Lemma 3.3.3, this reduces to maximizing

$$[\,\theta^T \int c(x)\, p_0(x)\, d\lambda(x) - \psi(\theta)\,]\ .$$

Assuming the maximum is achieved in $\theta_0 \in \Theta$, necessary conditions yield

$$\eta_i(\theta_0) = \int c_i(x)\, p_0(x)\, d\lambda(x)\ , \qquad i = 1, \cdots, m\ .$$

4.5 The residual and a convenient exponential family

In this section, we are interested in defining quantities which will provide a local measure of the quality of the projection–filter approximation. Compare equation (2.7) for the (square root of the) true density p_t, i.e.

$$d\sqrt{p_t} = \mathcal{P}_t(\sqrt{p_t})\, dt - \mathcal{Q}_t^0(\sqrt{p_t})\, dt + \sum_{k=1}^{d} \mathcal{Q}_t^k(\sqrt{p_t}) \circ dY_t^k\ , \tag{4.15}$$

and equation (4.11) for the (square root of the) projection–filter density $p_t^\pi = p(\cdot, \theta_t)$, i.e.

$$d\sqrt{p_t^\pi} = \Pi_{\theta_t} \circ \mathcal{P}_t(\sqrt{p_t^\pi})\, dt - \Pi_{\theta_t} \circ \mathcal{Q}_t^0(\sqrt{p_t^\pi})\, dt + \sum_{k=1}^{d} \Pi_{\theta_t} \circ \mathcal{Q}_t^k(\sqrt{p_t^\pi}) \circ dY_t^k\ . \tag{4.16}$$

Two steps are involved in using the projection–filter density p_t^π as an approximation of the true density p_t : We make a first approximation by evaluating the right–hand side of equation (4.15) at the current projection–filter density p_t^π and not at the true density p_t. Even with this approximation, the resulting coefficients $\mathcal{P}_t(\sqrt{p_t^\pi})$ and $\mathcal{Q}_t^k(\sqrt{p_t^\pi})$ for $k = 0, 1, \cdots, d$ would make the solution leave the manifold $S^{1/2}$, and we make a second approximation by projecting these coefficients on the linear space $L_{\sqrt{p_t^\pi}} S^{1/2}$ via the projection mapping Π_{θ_t}. In order to express the error occurring in the second approximation step at time t, we define the prediction residual operator $\mathcal{R}_t^\bullet$ and the correction residual operators $\mathcal{R}_t^k$ for $k = 0, 1, \cdots, d$ as follows :

$$\begin{aligned}\mathcal{R}_t^\bullet &:= \mathcal{P}_t - \Pi_{\theta_t} \circ \mathcal{P}_t \\ \mathcal{R}_t^k &:= \mathcal{Q}_t^k - \Pi_{\theta_t} \circ \mathcal{Q}_t^k \ .\end{aligned}$$

These operators, when applied to the square root of density $\sqrt{p_t^\pi} = \sqrt{p(\cdot, \theta_t)} \in S^{1/2}$ yield vectors of $L_2(\lambda)$. We call such vectors *projection residuals* : they give a local measure of the quality of the approximation involved in the projection filter. We can compute the norm of such vectors according to the norm $\|\cdot\|$ in $L_2(\lambda)$, and we define the prediction residual norm $r_t^\bullet$ and correction residual norms r_t^k for $k = 0, 1, \cdots, d$ as follows :

$$\begin{aligned}r_t^\bullet &:= \|\mathcal{R}_t^\bullet(\sqrt{p_t^\pi})\| \\ r_t^k &:= \|\mathcal{R}_t^k(\sqrt{p_t^\pi})\| \ .\end{aligned}$$

However, we are still missing a single measure of the local error resulting from the projection. We define below a single residual operator, only in the case where $\mathcal{R}_t^k = 0$ for all $t \geq 0$, and all $k = 1, \cdots, d$. In this case, we define the total residual operator $\mathcal{R}_t^*$ as :

$$\mathcal{R}_t^* := \mathcal{R}_t^\bullet - \mathcal{R}_t^0 \ ,$$

and the corresponding total residual norm r_t^* as :

$$r_t^* := \|\mathcal{R}_t^*(\sqrt{p_t^\pi})\| \ .$$

Notice that if in addition $\mathcal{R}_t^0 = 0$, then r_t^* reduces to $r_t^\bullet$. In the next section we will introduce manifolds $S_\bullet^{1/2}$ and $S_*^{1/2}$ for which such a definition is applicable. Now we try to give some intuition for the above definition. Suppose we replace in equations (4.15) and (4.16) the observation $\{Y_t\,, t \geq 0\}$ with some smooth process $\{u_t\,, t \geq 0\}$, e.g. a regularized approximation, i.e. we consider the equations

$$\frac{d}{dt}\sqrt{p_t} = \mathcal{P}_t(\sqrt{p_t}) - \mathcal{Q}_t^0(\sqrt{p_t}) + \sum_{k=1}^d \mathcal{Q}_t^k(\sqrt{p_t})\ \dot{u}_t^k \ , \tag{4.17}$$

and

$$\frac{d}{dt}\sqrt{p_t^\pi} = \Pi_{\theta_t} \circ \mathcal{P}_t(\sqrt{p_t^\pi}) - \Pi_{\theta_t} \circ \mathcal{Q}_t^0(\sqrt{p_t^\pi}) + \sum_{k=1}^d \Pi_{\theta_t} \circ \mathcal{Q}_t^k(\sqrt{p_t^\pi})\ \dot{u}_t^k \ . \tag{4.18}$$

In this case, we can define a single residual operator expressing the difference between the rate of change in the smooth Kushner–Stratonovich equation (4.17) and the rate of change in the smooth projection filter equation (4.18), i.e.

$$\mathcal{R}_t^u := \mathcal{R}_t^\bullet - \mathcal{R}_t^0 + \sum_{k=1}^d \mathcal{R}_t^k \, \dot{u}_t^k \ .$$

Of course, if we return to the original situation, e.g. letting the regularized approximation $\{u_t \, , \, t \geq 0\}$ converge to the observation $\{Y_t \, , \, t \geq 0\}$, there is no limit to the smooth residual operator $\mathcal{R}_t^u$, *unless* $\mathcal{R}_t^k = 0$ for all $t \geq 0$, and all $k = 1, \cdots, d$. In this case only, we define the total residual operator $\mathcal{R}_t^*$ as above.

Note that $\mathcal{R}_t^*$ is the residual error in the 'dt' term of the SDE describing the projection filter.

From now on, and throughout the chapter, we assume for simplicity that $h_t(x) = h(x)$ does not depend explicitly on time. This is necessary in order to define the simplifying *time invariant* exponential families $S_\bullet$ and S_* below.

4.5.1 The exponential families $S_\bullet$ and S_*

Now we can state the following

Theorem 4.5.1 *Assumptions (A), (B) and (C) on the coefficients f_t, a_t and h of the system (2.1) in force. Let $s := \mathrm{rank}\{h^1, \cdots, h^d, \frac{1}{2}|h|^2\} \leq d+1$. There exist s linearly independent functions $\{c_1, \cdots, c_s\}$ defined on $\mathbf{R}^n$, such that for all $x \in \mathbf{R}^n$*

$$\tfrac{1}{2}\, |h(x)|^2 = \sum_{i=1}^s \lambda_i^0 \, c_i(x) \ , \qquad h^k(x) = \sum_{i=1}^s \lambda_i^k \, c_i(x) \ , \tag{4.19}$$

for $k = 1, \cdots, d$. Remaining functions $\{c_{s+1}, \cdots, c_m\}$ are chosen such that

$$S_\bullet := \{p(\cdot, \theta) \, , \, \theta \in \Theta\} \ , \qquad p(x, \theta) := \exp[\theta^T c(x) - \psi(\theta)] \ ,$$

where $\Theta \subseteq \mathbf{R}^m$ is open, is an exponential family of probability densities.

Assume that, in addition to (A), (B) and (C), the coefficients f_t and a_t of the system (2.1), and the coefficients c of the exponential family $S_\bullet$ are such that :

$$E_{p(\cdot,\theta)}\{|\alpha_{t,\theta}|^2\} < \infty \ ,$$

holds for all $\theta \in \Theta$, and all $t \geq 0$, where the expression of $\alpha_{t,\theta}$ is given in (4.6).

Then, for the projection filter associated with the exponential family $S_\bullet$, the correction residual norms r_t^k are identically zero for all $t \geq 0$, and all $k = 0, 1, \cdots, d$, and the stochastic differential equation for the parameters reduces to :

$$d\theta_t = [g(\theta_t)]^{-1} \, E_{p(\cdot,\theta_t)}\{\mathcal{L}_t \, c\} \, dt - \lambda_\bullet^0 \, dt + \sum_{k=1}^d \lambda_\bullet^k \, dY_t^k \ , \quad \theta_0. \tag{4.20}$$

where for all $k = 0, 1, \cdots, d$ the m–dimensional vector $\lambda^k_\bullet$ is defined by

$$\lambda^k_\bullet = \begin{bmatrix} \lambda^k_1 \\ \vdots \\ \lambda^k_s \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

Under the assumptions on the coefficients, this equation has a unique solution, up to the a.s. positive exit time $\tau := \inf\{t > 0 \, : \, \theta_t \notin \Theta\}$.

PROOF : All the assumptions of Theorem 4.4.3 are satisfied, and therefore the solution of the stochastic differential equation for the projection filter with manifold $S^{1/2}_\bullet$ exists and is unique up to the a.s. positive exit time τ.

Next, we prove that the correction residual norms vanish. Indeed, it follows from (4.19) that

$$\begin{aligned} \mathcal{Q}^0_t(\sqrt{p(\cdot,\theta_t)}) &= \tfrac{1}{4}\,[|h|^2 - E_{p(\cdot,\theta_t)}\{|h|^2\}]\,\sqrt{p(\cdot,\theta_t)} \\ &= \tfrac{1}{2}\sum_{i=1}^s \lambda^0_i\,[c_i - E_{p(\cdot,\theta_t)}\{c_i\}]\,\sqrt{p(\cdot,\theta_t)} \;, \end{aligned}$$

and similarly

$$\begin{aligned} \mathcal{Q}^k_t(\sqrt{p(\cdot,\theta_t)}) &= \tfrac{1}{2}\,[h^k - E_{p(\cdot,\theta_t)}\{h^k\}]\,\sqrt{p(\cdot,\theta_t)} \\ &= \tfrac{1}{2}\sum_{i=1}^s \lambda^k_i\,[c_i - E_{p(\cdot,\theta_t)}\{c_i\}]\,\sqrt{p(\cdot,\theta_t)} \;, \end{aligned}$$

for $k = 1, \cdots, d$. We remark that

$$\tfrac{1}{2}\,[c_i - E_{p(\cdot,\theta_t)}\{c_i\}]\,\sqrt{p(\cdot,\theta_t)} = \tfrac{1}{2}\,[c_i - \eta_i(\theta_t)]\,\sqrt{p(\cdot,\theta_t)} = \frac{1}{2\sqrt{p(\cdot,\theta_t)}}\,\frac{\partial p(\cdot,\theta_t)}{\partial\theta_i} \;,$$

hence $\mathcal{Q}^k_t(\sqrt{p(\cdot,\theta_t)}) \in L_{\sqrt{p(\cdot,\theta_t)}} S^{1/2}$ for $k = 0, 1, \cdots, d$. Therefore, the projection does not modify these vectors since they already lie in the tangent space of $S^{1/2}$.

Finally, the equation for the parameters is obtained via straightforward calculations. Indeed, it follows from (4.19) that

$$E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\,|h|^2\,[c_j - \eta_j(\theta_t)]\} = \sum_{l=1}^s \lambda^0_l\,E_{p(\cdot,\theta_t)}\{c_l\,[c_j - \eta_j(\theta_t)]\} = \sum_{l=1}^s g_{jl}(\theta_t)\,\lambda^0_l \;,$$

hence

$$\sum_{j=1}^{m} g^{ij}(\theta_t)\ E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\,|h|^2\ [c_j - \eta_j(\theta_t)]\} = \sum_{j=1}^{m} g^{ij}(\theta_t)\ \sum_{l=1}^{s} g_{jl}(\theta_t)\ \lambda_l^0 = \sum_{l=1}^{s} \delta_{il}\ \lambda_l^0\ ,$$

and similarly

$$\sum_{j=1}^{m} g^{ij}(\theta_t)\ E_{p(\cdot,\theta_t)}\{h^k\ [c_j - \eta_j(\theta_t)]\} = \sum_{l=1}^{s} \delta_{il}\ \lambda_l^k\ ,$$

for all $k = 1, \cdots, d$. Substituting these expressions into the right–hand side of equation (4.14) yields

$$d\theta_t^i = [\sum_{j=1}^{m} g^{ij}(\theta_t)\ E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c_j\}]\ dt - [\sum_{l=1}^{s} \delta_{il}\ \lambda_l^0]\ dt + \sum_{k=1}^{d}\ [\sum_{l=1}^{s} \delta_{il}\ \lambda_l^k]\ dY_t^k\ .$$

In vector form, the above equation reads

$$d\theta_t = [g(\theta_t)]^{-1}\ E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c\}\, dt - \lambda_\bullet^0\, dt + \sum_{k=1}^{d}\ \lambda_\bullet^k\, dY_t^k\ .$$

This finishes the proof. □

What the above theorem shows is that the projection residuals are greatly simplified if we make use of the functions $\{h^1, \cdots, h^d, \frac{1}{2}|h|^2\}$ in the definition of the exponential manifold, i.e. if we choose the functions $\{c_1, \cdots, c_m\}$ in such a way that the functions $\{h^1, \cdots, h^d, \frac{1}{2}|h|^2\}$ are in span$\{c_1, \cdots, c_m\}$. Indeed, $\mathcal{R}_t^k(\sqrt{p_t^\pi}) = 0$ for all $t \geq 0$, and all $k = 0, 1, \cdots, k$, whereas

$$\frac{1}{\sqrt{p_t^\pi}}\ \mathcal{R}_t^\bullet(\sqrt{p_t^\pi}) = \tfrac{1}{2}\,\frac{\mathcal{L}_t^*\, p_t^\pi}{p_t^\pi} - \tfrac{1}{2}\,[c - \eta(\theta_t)]^T\ [g(\theta_t)]^{-1}\ E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c\}\ . \tag{4.21}$$

The diffusion coefficient in the stochastic differential equation (4.20) for the parameters θ_t is constant. This implies that (4.20) can be seen as either an Itô or a MFS stochastic differential equation, so that it satisfies the formal rules of calculus. Moreover, for the numerical solution of such an equation, the simpler Euler scheme coincides with the Milshtein scheme, which is a strongly convergent scheme of order 1, see [38].

Notice also that we have still some freedom left, and we may wonder whether one can use this to select m and the remaining functions $\{c_{s+1}, \cdots, c_m\}$ in order to reduce the total residual norm $r_t^* = r_t^\bullet$. However, a great prudence is needed, because the filter may become complicated and numerical problems may arise. See examples on the cubic sensor in Section 4.6. In general, a trade–off is necessary in order to obtain an accurate, but still not too involved, exponential family and the associated projection filter.

Similarly to the Theorem 4.5.1 above, we have the following

Theorem 4.5.2 *Assumptions (A), (B) and (C) on the coefficients f_t, a_t, and h of the system (2.1) in force. Let $s := \mathrm{rank}\{h^1, \cdots, h^d\} \leq d$. There exist s linearly independent functions $\{c_1, \cdots, c_s\}$ defined on $\mathbf{R}^n$, such that for all $x \in \mathbf{R}^n$*

$$h^k(x) = \sum_{i=1}^{s} \lambda_i^k \, c_i(x) \ ,$$

for $k = 1, \cdots, d$. Remaining functions $\{c_{s+1}, \cdots, c_m\}$ are chosen such that

$$S_* := \{p(\cdot, \theta) \, , \, \theta \in \Theta\} \ , \qquad p(x, \theta) := \exp[\theta^T c(x) - \psi(\theta)] \ ,$$

where $\Theta \subseteq \mathbf{R}^m$ is open, is an exponential family of probability densities.

Assume that, in addition to (A), (B) and (C), the coefficients f_t and a_t of the system (2.1), and the coefficients c of the exponential family S_* are such that :

$$E_{p(\cdot,\theta)}\{|\alpha_{t,\theta}|^2\} < \infty \ ,$$

holds for all $\theta \in \Theta$, and all $t \geq 0$, where the expression of $\alpha_{t,\theta}$ is given in (4.6).

Then, for the projection filter associated with the exponential family S_*, the correction residual norms r_t^k are identically zero for all $t \geq 0$, and all $k = 1, \cdots, d$, and the stochastic differential equation for the parameters reduces to :

$$\begin{aligned} d\theta_t \ &= \ [g(\theta_t)]^{-1} \, E_{p(\cdot,\theta_t)}\{\mathcal{L}_t \, c\} \, dt \\ &- \ [g(\theta_t)]^{-1} \, E_{p(\cdot,\theta_t)}\{\tfrac{1}{2} \, |h|^2 \, [c - \eta(\theta_t)]\} \, dt + \sum_{k=1}^{d} \lambda_*^k \, dY_t^k \ , \quad \theta_0 \ , \end{aligned} \tag{4.22}$$

where for all $k = 1, \cdots, d$ the m–dimensional vector λ_*^k is defined by

$$\lambda_*^k = \begin{bmatrix} \lambda_1^k \\ \vdots \\ \lambda_s^k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ .$$

Under the assumptions on the coefficients, this equation has a unique solution, up to the a.s. positive exit time $\tau := \inf\{t > 0 \, : \, \theta_t \notin \Theta\}$.

The proof is analogous to the proof of Theorem 4.5.1, and is therefore omitted.

In this case, $\mathcal{R}_t^k(\sqrt{p_t^\pi}) = 0$ for all $t \geq 0$, and all $k = 1, \cdots, d$, whereas

$$\frac{1}{\sqrt{p_t^\pi}}\,\mathcal{R}_t^\bullet(\sqrt{p_t^\pi}) = \tfrac{1}{2}\,\frac{\mathcal{L}_t^*\,p_t^\pi}{p_t^\pi} - \tfrac{1}{2}\,[c - \eta(\theta_t)]^T\,[g(\theta_t)]^{-1}\,E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\,c\}\ , \tag{4.23}$$

and

$$\begin{aligned}\frac{1}{\sqrt{p_t^\pi}}\,\mathcal{R}_t^0(\sqrt{p_t^\pi}) \;&=\; \tfrac{1}{4}\,[|h|^2 - E_{p(\cdot,\theta_t)}\{|h|^2\}] \\ &\quad - \tfrac{1}{2}\,[c - \eta(\theta_t)]^T\,[g(\theta_t)]^{-1}\,E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\,|h|^2\,[c - \eta(\theta_t)]\}\ .\end{aligned} \tag{4.24}$$

4.5.2 The case of discrete–time observations

We conclude this section by presenting the effect of choosing the exponential family $S_\bullet$, in the case of a nonlinear filtering problem with discrete–time observations. In this model, the state process is as in equation (2.1), i.e.

$$dX_t = f_t(X_t)\,dt + \sigma_t(X_t)\,dW_t\ ,$$

but only discrete–time observations are available

$$z_n = h(X_{t_n}) + v_n\ ,$$

at times $0 = t_0 < t_1 < \cdots < t_n < \cdots$ regularly sampled, where $\{v_n\,,\, n \geq 0\}$ is a Gaussian white noise sequence independent of $\{X_t\,,\, t \geq 0\}$.

The nonlinear filtering problem consists in finding the conditional density $p_n(x)$ of the state X_{t_n} given the observations up to time t_n, i.e. such that $P[X_{t_n} \in dx \mid \mathcal{Z}_n] = p_n(x)\,dx$, where $\mathcal{Z}_n := \sigma(z_0, \cdots, z_n)$. We define also the prediction conditional density $p_n^-(x)\,dx = P[X_{t_n} \in dx \mid \mathcal{Z}_{n-1}]$. The sequence $\{p_n\,,\, n \geq 0\}$ satisfies a recurrent equation, and the transition from p_{n-1} to p_n is decomposed in two steps, as explained in [32], [47] :

Prediction step Between time t_{n-1} and t_n, we solve the Fokker–Planck equation

$$\frac{\partial p_t^n}{\partial t} = \mathcal{L}_t^*\,p_t^n\ , \qquad p_{t_{n-1}}^n = p_{n-1}\ .$$

The solution at final time t_n defines the prediction conditional density $p_n^- = p_{t_n}^n$.

Correction step At time t_n, the observation z_n is combined with the prediction conditional density p_n^- via the Bayes rule

$$p_n(x) = c_n\;\Psi_n(x)\;p_n^-(x)\ , \tag{4.25}$$

where c_n is a normalizing constant, and $\Psi_n(x)$ denotes the likelihood function for the estimation of X_{t_n} based on the observation z_n only, i.e.

$$\Psi_n(x) := \exp\{\, -\tfrac{1}{2}|z_n - h(x)|^2 \,\} \ . \tag{4.26}$$

If we use the exponential family $S_\bullet$ defined above, then we obtain the projection filter density $p(\cdot, \theta_n)$, and the transition from θ_{n-1} to θ_n is also decomposed in two steps :

Prediction step Between time t_{n-1} and t_n, we solve the ODE

$$g(\theta^n_t)\, \dot\theta^n_t = E_{p(\cdot,\theta^n_t)}\{\mathcal{L}_t\, c\} \ , \qquad \theta^n_{t_{n-1}} = \theta_{n-1} \ .$$

The solution at final time t_n defines the prediction parameters $\theta^-_n = \theta^n_{t_n}$.

Correction step Substituting the approximation $p(\cdot, \theta^-_n)$ into formula (4.25), we observe that the resulting density does not leave the exponential family $S_\bullet$. Indeed, it follows from (4.19) and (4.26) that

$$\begin{aligned}
\Psi_n(x) &= \exp\{\, -\tfrac{1}{2}\, |h(x)|^2 + \sum_{k=1}^{d} h^k(x)\, z^k_n - \tfrac{1}{2}\, |z_n|^2 \,\} \\
&= \exp\{\, -\sum_{l=1}^{s} \lambda^0_l\, c_l(x) + \sum_{l=1}^{s} [\sum_{k=1}^{d} \lambda^k_l\, z^k_n]\; c_l(x) - \tfrac{1}{2}\, |z_n|^2 \,\} \ ,
\end{aligned}$$

and the parameters are updated according to the formula

$$\theta_n = \theta^-_n - \lambda^0_\bullet + \sum_{k=1}^{d} \lambda^k_\bullet\, z^k_n \ ,$$

which is *exact*.

4.6 Exponential projection filters for the cubic sensor

We consider as an application of the exponential projection filter the explicit formula for the cubic sensor, see also [29]. We consider the scalar system

$$\begin{aligned}
dX_t &= \sigma\, dW_t \\
dY_t &= X^3_t\, dt + dV_t \ ,
\end{aligned}$$

with the usual independence assumptions for the standard Brownian motions $\{W_t\,,\, t \geq 0\}$ and $\{V_t\,,\, t \geq 0\}$ and where σ is a real constant. This system is interesting for several reasons. First, the simplicity of the state process. Secondly, the infinite–dimensionality of the optimal filter for the cubic sensor ensures that we are really facing a problem of approximating an infinite–dimensional filter by a finite–dimensional one. The fact that the optimal filter for the cubic sensor is infinite dimensional was proved in [30].

Let us apply the projection filter to this system using different exponential families in order to illustrate how the filter depends on the manifold.

4.6.1 The six dimensional exponential projection filter

We choose the manifold S according to Theorem 4.5.1, i.e.

$$\begin{aligned} S &= S_\bullet = \{p(\cdot,\theta)\,,\, \theta \in \Theta\}\ , \\ p(x,\theta) &= \exp[\theta_1\, x + \theta_2\, x^2 + \theta_3\, x^3 + \theta_4\, x^4 + \theta_5\, x^5 + \theta_6\, x^6 - \psi(\theta)]\ , \end{aligned}$$

where Θ is open in $\mathbf{R}^6$ and $\theta_6 < 0$, for all $\theta \in \Theta$.

We notice that $h(x) = x^3$ and $\frac{1}{2}\,|h(x)|^2 = \frac{1}{2}\,x^6$, hence

$$\lambda^0_\bullet = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}\ , \qquad \lambda_\bullet := \lambda^1_\bullet = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\ .$$

On the other hand, $c_j(x) = x^j$, for $j = 1, \cdots, 6$, hence

$$\mathcal{L}\, c_j(x) = \tfrac{1}{2}\,\sigma^2\, \frac{\partial^2 c_j(x)}{\partial x^2} = \begin{cases} \frac{1}{2}\,\sigma^2\, j\,(j-1)\, x^{j-2}\ , & \text{for } j = 2, \cdots, 6 \\ 0\ , & \text{for } j = 1 \end{cases}$$

and therefore

$$E_{p(\cdot,\theta)}\{\mathcal{L}\, c_j\} = \begin{cases} \frac{1}{2}\,\sigma^2\, j\,(j-1)\, \eta_{j-2}(\theta)\ , & \text{for } j = 2, \cdots, 6 \\ 0\ , & \text{for } j = 1 \end{cases}$$

which requires the evaluation of $\eta_0(\theta), \cdots, \eta_4(\theta)$. We define

$$\gamma_\bullet(\theta) := \tfrac{1}{2}\,\sigma^2 \begin{bmatrix} 0 \\ 2\,\eta_0(\theta) \\ 6\,\eta_1(\theta) \\ 12\,\eta_2(\theta) \\ 20\,\eta_3(\theta) \\ 30\,\eta_4(\theta) \end{bmatrix} = E_{p(\cdot,\theta)}\{\mathcal{L}\,c\} \ .$$

Finally, the entries of the Fisher information matrix $(g_{ij}(\theta))$ are obtained according to (3.6), i.e.

$$g_{ij}(\theta) = \eta_{i+j}(\theta) - \eta_i(\theta)\,\eta_j(\theta) \ , \qquad i,j = 1,\cdots,6$$

which requires the evaluation of $\eta_1(\theta), \cdots, \eta_{12}(\theta)$. However, $\eta_0(\theta) = 1$ and it follows from Lemma 3.3.3 that only $\eta_1(\theta), \cdots, \eta_5(\theta)$ need to be evaluated, since $\eta_6(\theta), \cdots, \eta_{12}(\theta)$ can be obtained according to (3.5).

The stochastic differential equation (4.20) for the parameters reduces to

$$d\theta_t = [g(\theta_t)]^{-1}\,\gamma_\bullet(\theta_t)\,dt - \lambda^0_\bullet\,dt + \lambda_\bullet\,dY_t \ .$$

The equation (4.21) for the prediction residual reduces to

$$\frac{1}{\sqrt{p^\pi_t}}\,\mathcal{R}^\bullet_t(\sqrt{p^\pi_t}) = \tfrac{1}{2}\,\frac{\mathcal{L}^*\,p^\pi_t}{p^\pi_t} - \tfrac{1}{2}\,[c - \eta(\theta_t)]^T\,[g(\theta_t)]^{-1}\,\gamma_\bullet(\theta_t) \ ,$$

from which the total residual norm $r^*_t = r^\bullet_t$ can be easily computed.

Finally, we indicate a quantity which can be used to estimate the state of the system at time t. It is well known that, if the conditional density p_t is available, then the best (minimum–variance) estimator of X_t is the conditional expectation

$$\widehat{X}_t := E_{p_t}\{x\} = \int x\,p_t(x)\,d\lambda(x) \ .$$

As we can rely only on the approximated density $p(\cdot,\theta_t)$, we shall consider, as an estimate of the state, the expectation w.r.t. this approximated density :

$$\eta_1(\theta_t) = E_{p(\cdot,\theta_t)}\{x\} = \int x\,p(x,\theta_t)\,d\lambda(x) \ .$$

4.6.2 The four dimensional exponential projection filter

In this Section we choose the manifold S according to Theorem 4.5.2, i.e.

$$S = S_* = \{p(\cdot,\theta)\,,\,\theta \in \Theta\}\,, \qquad p(x,\theta) = \exp[\theta_1\,x + \theta_2\,x^2 + \theta_3\,x^3 + \theta_4\,x^4 - \psi(\theta)]\ ,$$

where $\Theta \subseteq \mathbf{R}^4$ is open and $\theta_4 < 0$, for all $\theta \in \Theta$.

We notice that $h(x) = x^3$, hence

$$\lambda_* := \lambda_*^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

On the other hand, $c_j(x) = x^j$, for $j = 1, \cdots, 4$, hence

$$\mathcal{L}\, c_j(x) = \tfrac{1}{2}\,\sigma^2\, \frac{\partial^2 c_j(x)}{\partial x^2} = \begin{cases} \tfrac{1}{2}\,\sigma^2\, j\,(j-1)\, x^{j-2} \;, & \text{for } j = 2, \cdots, 4 \\ \\ 0 \;, & \text{for } j = 1 \end{cases}$$

and therefore

$$E_{p(\cdot,\theta)}\{\mathcal{L}\, c_j\} = \begin{cases} \tfrac{1}{2}\,\sigma^2\, j\,(j-1)\, \eta_{j-2}(\theta) \;, & \text{for } j = 2, \cdots, 4 \\ \\ 0 \;, & \text{for } j = 1 \end{cases}$$

which requires the evaluation of $\eta_0(\theta), \cdots, \eta_2(\theta)$. We define

$$\gamma_*(\theta) := \tfrac{1}{2}\,\sigma^2 \begin{bmatrix} 0 \\ 2\,\eta_0(\theta) \\ 6\,\eta_1(\theta) \\ 12\,\eta_2(\theta) \end{bmatrix} = E_{p(\cdot,\theta)}\{\mathcal{L}\, c\} \;. \tag{4.27}$$

Similarly, we notice that $\tfrac{1}{2}\,|h(x)|^2 = \tfrac{1}{2}\, x^6$, hence

$$\tfrac{1}{2}\,|h(x)|^2\,[c_j(x) - \eta_j(\theta)] = \tfrac{1}{2}\,[x^{6+j} - x^6\, \eta_j(\theta)] \;, \qquad j = 1, \cdots, 4$$

and

$$E_{p(\cdot,\theta)}\{\tfrac{1}{2}\,|h|^2\,[c_j - \eta_j(\theta)]\} = \tfrac{1}{2}\,[\eta_{6+j}(\theta) - \eta_6(\theta)\eta_j(\theta)] \;, \qquad j = 1, \cdots, 4$$

which requires the evaluation of $\eta_1(\theta), \cdots, \eta_4(\theta)$ and $\eta_6(\theta), \cdots, \eta_{10}(\theta)$. We define

$$\gamma_*^0(\theta) := \tfrac{1}{2} \begin{bmatrix} \eta_7(\theta) - \eta_6(\theta)\,\eta_1(\theta) \\ \eta_8(\theta) - \eta_6(\theta)\,\eta_2(\theta) \\ \eta_9(\theta) - \eta_6(\theta)\,\eta_3(\theta) \\ \eta_{10}(\theta) - \eta_6(\theta)\,\eta_4(\theta) \end{bmatrix} = E_{p(\cdot,\theta)}\{\tfrac{1}{2}\,|h|^2\,[c - \eta(\theta)]\} \;. \tag{4.28}$$

Finally, the entries of the Fisher information matrix $(g_{ij}(\theta))$ are obtained according to (3.6), i.e.

$$g_{ij}(\theta) = \eta_{i+j}(\theta) - \eta_i(\theta)\,\eta_j(\theta) \;, \qquad i,j = 1, \cdots, 4$$

which requires the evaluation of $\eta_1(\theta), \cdots, \eta_8(\theta)$. However, $\eta_0(\theta) = 1$ and it follows from Lemma 3.3.3 that only $\eta_1(\theta), \cdots, \eta_3(\theta)$ need to be evaluated, since $\eta_4(\theta), \cdots, \eta_{10}(\theta)$ can be obtained according to (3.5).

The stochastic differential equation (4.22) for the parameters reduces to

$$d\theta_t = [g(\theta_t)]^{-1}\, \gamma_*(\theta_t)\, dt - [g(\theta_t)]^{-1}\, \gamma_*^0(\theta_t)\, dt + \lambda_*\, dY_t \; . \tag{4.29}$$

The equations (4.23) and (4.24) for the prediction and correction residuals reduce to

$$\frac{1}{\sqrt{p_t^\pi}}\, \mathcal{R}_t^\bullet(\sqrt{p_t^\pi}) = \tfrac{1}{2}\, \frac{\mathcal{L}^*\, p_t^\pi}{p_t^\pi} - \tfrac{1}{2}\, [c - \eta(\theta_t)]^T\; [g(\theta_t)]^{-1}\, \gamma_*(\theta_t) \; ,$$

and

$$\frac{1}{\sqrt{p_t^\pi}}\, \mathcal{R}_t^0(\sqrt{p_t^\pi}) = \tfrac{1}{4}\, [x^6 - \eta_6(\theta_t)] - \tfrac{1}{2}\, [c - \eta(\theta_t)]^T\; [g(\theta_t)]^{-1}\, \gamma_*^0(\theta_t) \; ,$$

respectively, from which the total residual norm r_t^* can be easily computed.

Finally, as in Section 4.6.1 our approximation of the minimum variance estimate of the state at time t is the first expectation parameter $\eta_1(\theta_t)$. We conclude by observing that the filter given in this section can be implemented via a numerical scheme involving numerical–integration techniques. Such a scheme has been written as a Fortran program, yielding simulations that we describe in the next section.

4.7 Numerical simulations for the cubic sensor

In this section we present a numerical scheme which was used to implement the projection filter derived in Section 4.6.2, and we present also simulation results based on this numerical scheme. From the previous discussion, we need to compute the moments $\eta_1, \cdots, \eta_{10}$ up to order ten, but according to Lemma 3.3.3, these moments can be computed from the first three moments $\eta_1, \cdots, \eta_3$ only by using the recursion formula (3.5).

We applied a Euler scheme to solve the stochastic differential equation (4.29) numerically. Since the diffusion coefficient in this equation is constant, the Euler scheme coincides with the Milshtein scheme, and hence the error is of order Δ, where Δ is the chosen time step. In general, if the diffusion coefficient would also depend on the state θ then the error would be of order $\sqrt{\Delta}$ only. For a detailed treatment of numerical methods for stochastic differential equations, see [38].

We outline the main steps of the algorithm :

(i) Let an initial θ_0 be given. Choose a time step Δ and set $t = 0$.

(ii) Assign $\theta := \theta_0$.

(iii) Compute numerically the integral

$$I(\theta) := \exp[\psi(\theta)] = \int_{-\infty}^{+\infty} \exp[\theta_1\, x + \theta_2\, x^2 + \theta_3\, x^3 + \theta_4\, x^4]\; d\lambda(x)\ .$$

(iv) Compute the three following integrals, so as to obtain the first three expectation parameters :

$$\begin{aligned} \eta_i(\theta) &= E_{p(\cdot,\theta)}\{x^i\} \\ &= \frac{1}{I(\theta)} \int_{-\infty}^{+\infty} x^i\ \exp[\theta_1\, x + \theta_2\, x^2 + \theta_3\, x^3 + \theta_4\, x^4]\; d\lambda(x)\ , \end{aligned}$$

for $i = 1, 2, 3$.

(v) Compute the higher order moments $\eta_4, \cdots, \eta_{10}$ via the algebraic recursion formula given in (3.5).

(vi) Substitute the above quantities in equations (4.27) and (4.28), so as to obtain the coefficients $\gamma_*(\theta)$ and $\gamma_*^0(\theta)$ respectively.

(vii) Compute the Fisher information matrix

$$g_{ij}(\theta) = E_{p(\cdot,\theta)}\{x^i x^j\} - \eta_i\eta_j = \eta_{i+j} - \eta_i\eta_j\ , \qquad i, j = 1, \cdots, 4\ .$$

(viii) Invert $(g_{ij}(\theta))$ so as to obtain $(g^{ij}(\theta))$.

(ix) Collect the new observation $Y_{t+\Delta}$ at time $t + \Delta$ (here a discretization scheme is needed), and let $\Delta Y = Y_{t+\Delta} - Y_t$.

(x) Compute the approximate variation $\Delta\theta$ of the canonical parameters between times t and $t + \Delta$, according to the simple Euler scheme

$$\Delta\theta = [g(\theta)]^{-1}\, \gamma_*(\theta)\, \Delta - [g(\theta)]^{-1}\, \gamma_*^0(\theta)\, \Delta + \lambda_*\, \Delta Y\ .$$

(xi) Assign $\theta := \theta + \Delta\theta$ and $t := t + \Delta$.

(xii) Start again from point **(iii)**.

As noticed in step **(v)**, all we need is to compute the integrals

$$\int_{-\infty}^{+\infty} x^i\ \exp[\theta_1\, x + \theta_2\, x^2 + \theta_3\, x^3 + \theta_4\, x^4]\; d\lambda(x)\ , \qquad i = 0, \cdots, 3\ .$$

We used routines from the scientific library **NAG** for this purpose.

Once a numerical approximation of the projection filter parameters θ_t has been computed, we can compare the corresponding density $p_t^\pi = p(\cdot, \theta_t)$ to the solution p_t of the Kushner–Stratonovich equation, i.e. to the optimal filter density. Actually, a numerical approximation of p_t was used, based on a discretization of the state space with approximately 400 grid points, and on numerical techniques for the solution of stochastic differential equations, see [38] and [24].

The comparison between numerical approximations of the densities p_t^π and p_t can be done qualitatively, based on graphical outputs, or we can compute (a numerical approximation of) some distance, such as the Kullback–Leibler information

$$K(p_t, p_t^\pi) := \int \log \frac{p_t(x)}{p_t^\pi(x)} \, p_t(x) \, d\lambda(x) \ ,$$

the Hellinger distance

$$d(p_t, p_t^\pi) := \int (\sqrt{p_t^\pi(x)} - \sqrt{p_t(x)} \,)^2 \, d\lambda(x) = 2 \, [\, 1 - \int \sqrt{p_t^\pi(x)} \, \sqrt{p_t(x)} \, d\lambda(x) \,] \ ,$$

etc. We can also compute an approximation of the total residual norm

$$r_t^* := \| \mathcal{R}_t^\bullet(\sqrt{p_t^\pi}) - \mathcal{R}_t^0(\sqrt{p_t^\pi}) \| \ ,$$

which depends only on the projection filter density. As remarked in Section 4.6.2, the remaining correction residual norms r_t^k vanish for all $t \geq 0$ and all $k = 1, \cdots, d$. Moreover, to compute the total residual norm r_t^* we still need to evaluate only the first three moments.

We begin with some general remarks about our simulation results. These results show that the projection filter density is usually very close to the optimal filter density, when the latter is not too sharp (i.e. not too close to a Dirac mass). What would be missing in a Gaussian assumed–density filter or in an extended Kalman filter is the possibility to allow bimodality in the filter density. As the fourth degree exponential family allows such bimodality, in principle the optimal–filter density could be approximated at least qualitatively by a density in this family. This was actually observed in our simulations.

Moreover, we can have an a posteriori indication of the accuracy of the projection filter from the graphical representation of the total residual norm as a function of time. Indeed, there are time instants where the optimal–filter density and the projection–filter density are quite different, but these are exactly the time instants where the total residual norm exhibits large values. An additional observation that we could make on our simulations is that after a reasonably–small time the total residual norm returns towards zero, and correspondingly the projection–filter density is again very close to the optimal–filter density. To summarize, there are some differences, but they are limited in time, and do not seem to affect the global behaviour of the projection filter.

On time intervals where the true state is far from the singular point $x = 0$ of the observation function $h(x) = x^3$, experience shows that the smaller the observation noise, the sharper and higher are the peaks of the total residual norm. Notice first that if the observation noise is small, then on such time intervals the optimal–filter densities are concentrated around the true–state trajectory, i.e. are tracking accurately a very irregular trajectory. As a result, the difference between the mean value of the optimal–filter density and the mean value of the projection–filter density has to be really small, i.e. smaller than the variance of the optimal–filter density, to guarantee that the Hellinger distance between the optimal–filter density and the projection–filter density is not too large. This is reflected in the fast dynamics of both the Kushner–Stratonovich equation, and the equation for the projection–filter parameters, and makes the numerical implementation of the projection filter difficult when the observation noise is small.

In the following we discuss the simulations in detail, and we present some graphical outputs which illustrate our general remarks. In the two scalar examples below, the variance R of the observation noise does not satisfy $R = 1$. However, the formulas given in this chapter could easily be adapted to this more general situation.

Example 1 : We present here a first simulation of the fourth degree exponential projection filter based on the following data :

(unnormalized) initial density	$\exp[-\frac{1}{2}\,x^2 - \frac{1}{4}\,x^4]$
variance Q of the state noise	1
variance R of the observation noise	0.16
time step Δ	0.02
final time	10

In this first example we are mainly concerned in showing that our choice of the fourth degree exponential family is appropriate. Visualizing the time evolution of both the optimal–filter density and the projection–filter density was made possible with the software **ZPB** developed at INRIA. We observed that qualitatively the projection filter was good, as the two densities had roughly the same shape at every time instant. In this section we display the two densities at three time instants. We start by Figures 4.5 and 4.6 which show the true state and the estimate (mean value) provided by the *projection–filter density* respectively, as functions of time. This estimate is not accurate because on this simulation the true state stays most of the time around the singular point $x = 0$ of the observation function. Indeed, Figures 4.7 and 4.8 show that the mean value of the *optimal–filter density* does not provide an accurate estimate of the true state either. We are also interested in comparing the projection filter with the optimal filter, and not only with the true state. In this respect, Figures 4.1 and 4.3, show that the two filter estimates agree surprisingly well. Notice also the behaviour of the total residual norm in Figures 4.2 and 4.4 : the time instants where the two filter estimates are significantly different are characterized by large peaks in the total residual norm. This kind of simulation, where the conditional density is concentrated around the singular point of the observation function, is important because it is in such situations that Gaussian assumed–density filters and extended Kalman filters would usually fail. The shape of the density is quickly varying, becoming often bimodal and asymmetric, so that a Gaussian family is definitely not a good choice to base a finite–dimensional filtering on. We make this evident by displaying the optimal–filter and the projection–filter densities at different time instants, in Figures 4.9, 4.10, 4.11, 4.12, 4.13, and 4.14.

Example 2 : The second example is based on the following data :

(unnormalized) initial density	$\exp[-\frac{1}{2}\,(x-\frac{3}{4})^2 - \frac{1}{4}\,(x-\frac{3}{4})^4]$
variance Q of the state noise	1
variance R of the observation noise	9
time step Δ	0.005
final time	10

We begin by comparing the true state with the estimate (mean value) provided by the projection filter density. This is illustrated in Figures 4.19 and 4.20. It is clear from this graphical output that the state is not estimated accurately, and this is due to the fact that we have a large observation noise. Anyway, this is the case also for the optimal filter, as we can see in Figures 4.21 and 4.22. Nonetheless, our main concern is in the comparison between the projection filter and the optimal filter. This comparison is provided by Figures 4.15 and 4.17. The projection filter and the optimal filter estimates agree surprisingly well, and the time instants where they are significantly different are characterized by peaks of the total residual norm, which is shown in Figures 4.16 and 4.18. Finally, we remark that the numerical integrations involved in the implementation of the numerical scheme for the projection filter resulted in a large computational time. Indeed, the software *ZPB* of IRISA resulted to be quicker from a computational point of view, even though it employs a much larger number of parameters.

Figure 4.1: Mean projection filter and mean optimal filter between 0 and 5.

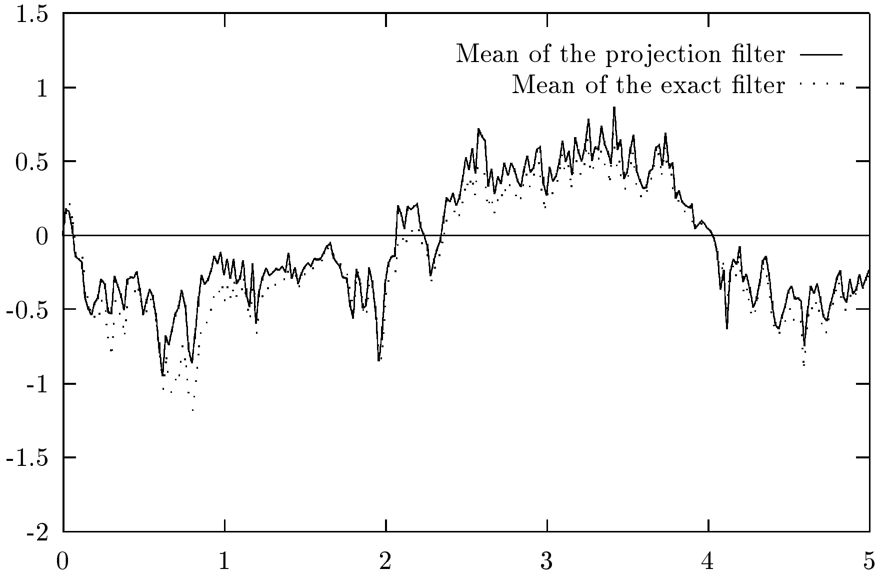


Figure 4.2: Projection residual between 0 and 5.

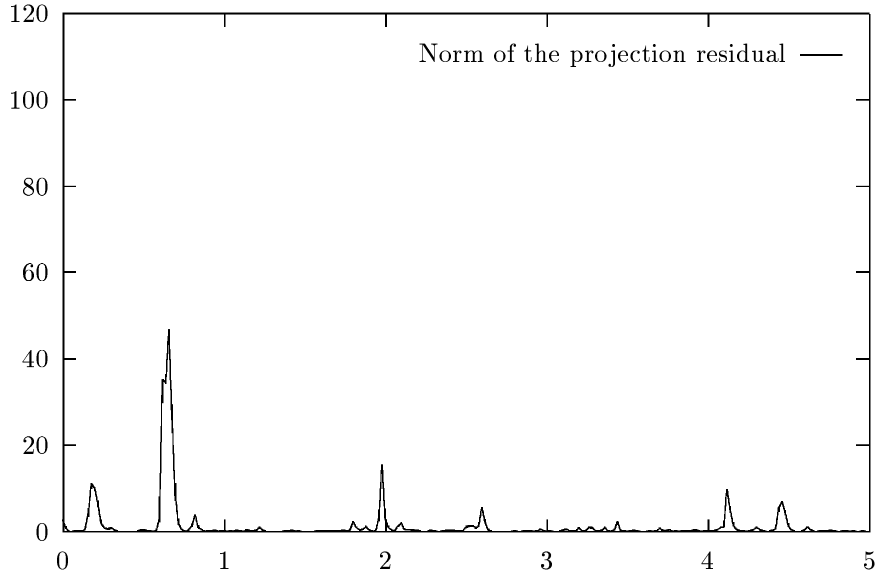


Figure 4.3: Mean projection filter and mean optimal filter between 5 and 10.

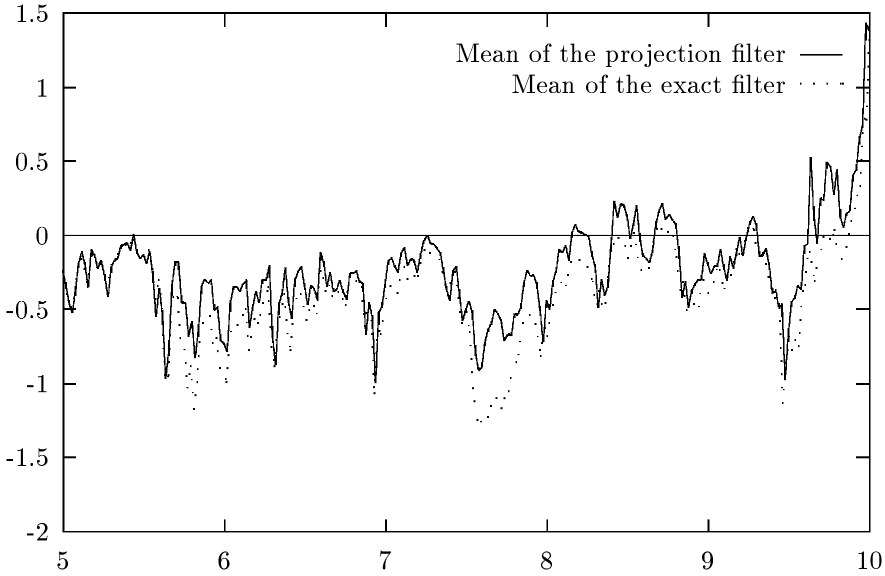


Figure 4.4: Projection residual between 5 and 10.

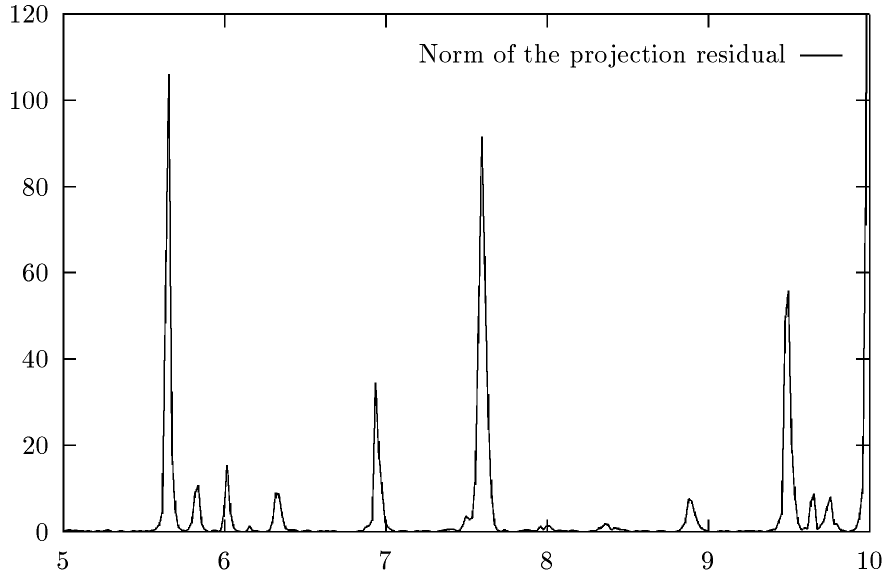


Figure 4.5: True state and mean from the projection filter between 0 and 5.

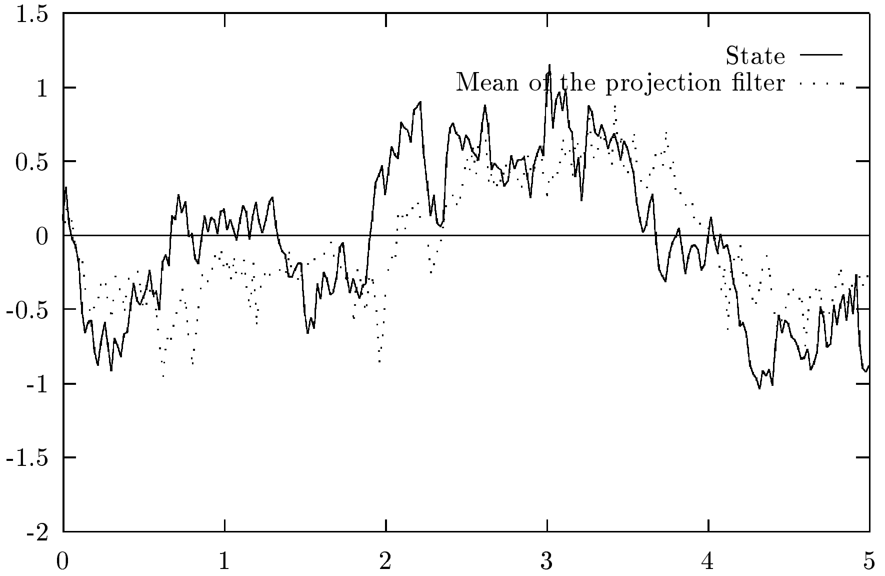


Figure 4.6: True state and mean from the projection filter between 5 and 10.

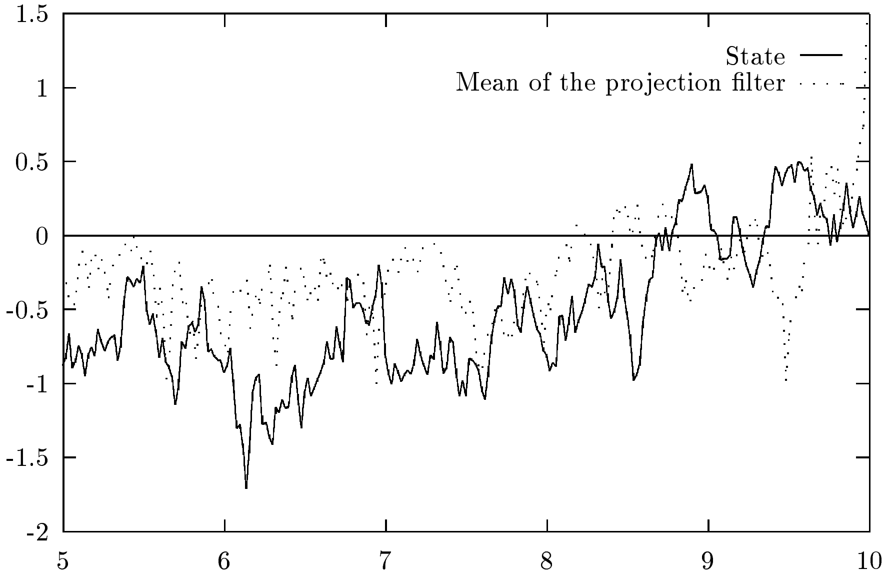


Figure 4.7: True state and mean from the optimal filter between 0 and 5.

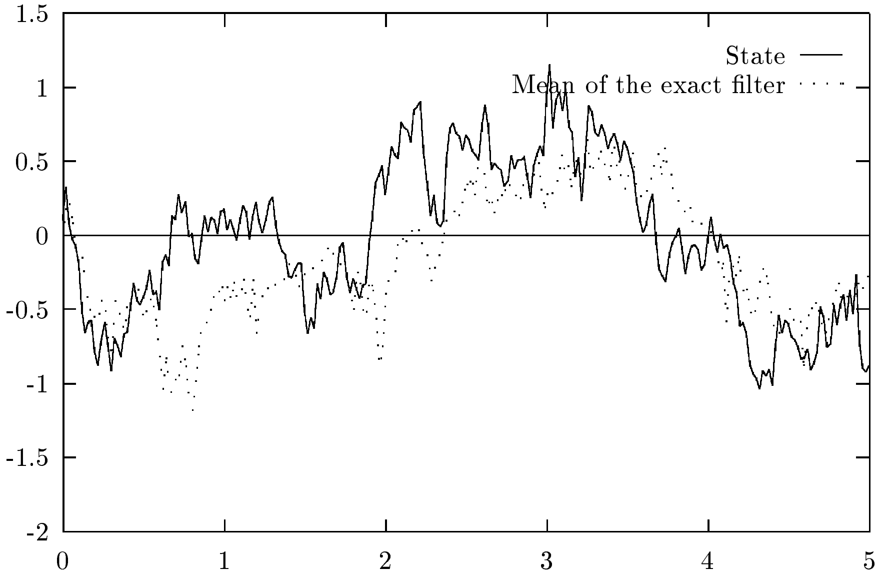


Figure 4.8: True state and mean from the optimal filter between 0 and 5.

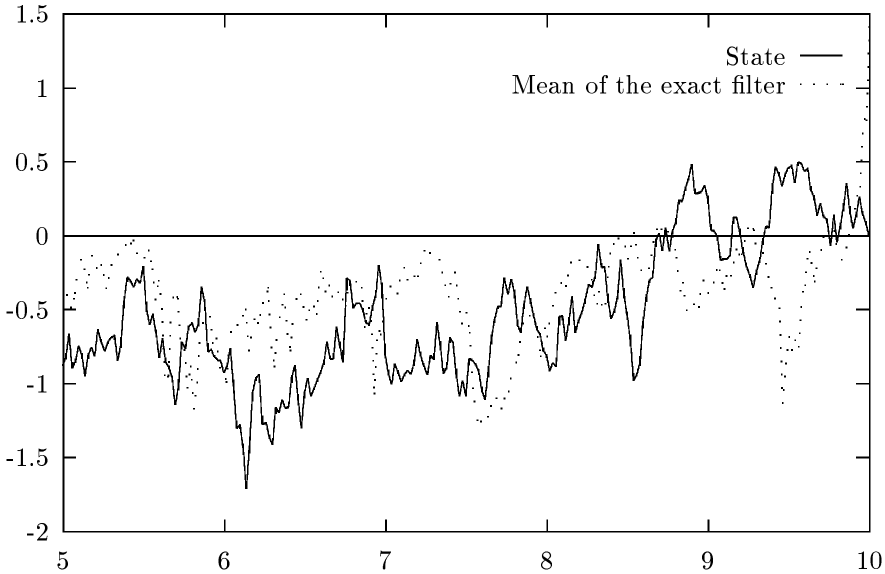


Figure 4.9: Optimal filter density at 3.70.

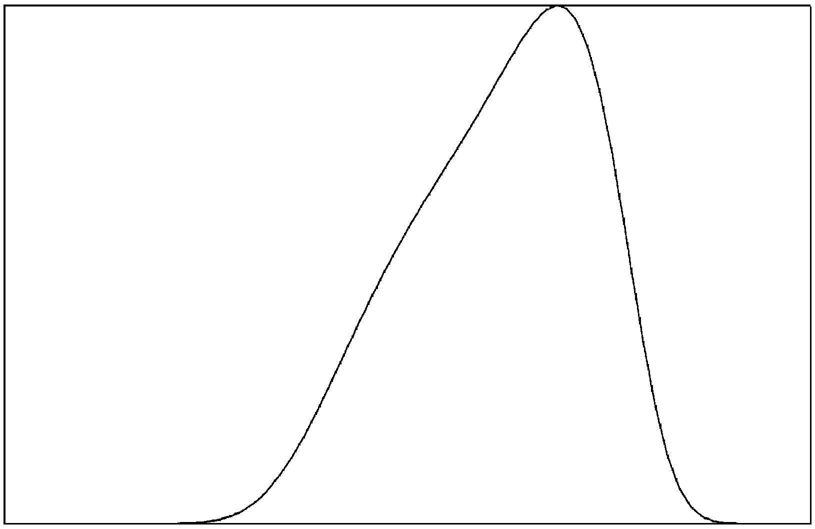


Figure 4.10: Projection filter density at 3.70.

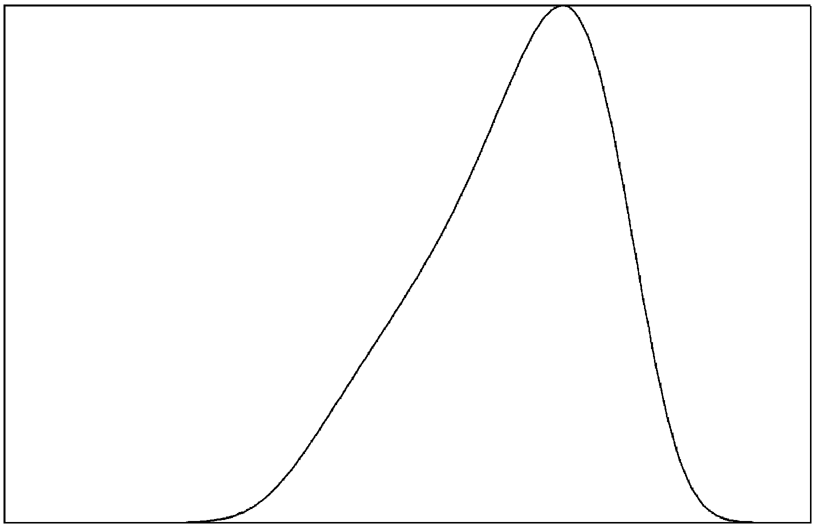


Figure 4.11: Optimal filter density at 4.12.

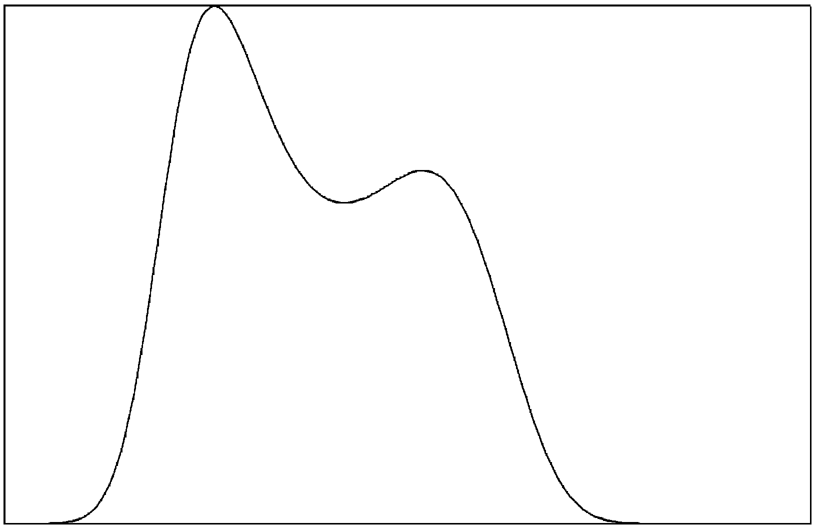


Figure 4.12: Projection filter density at 4.12.

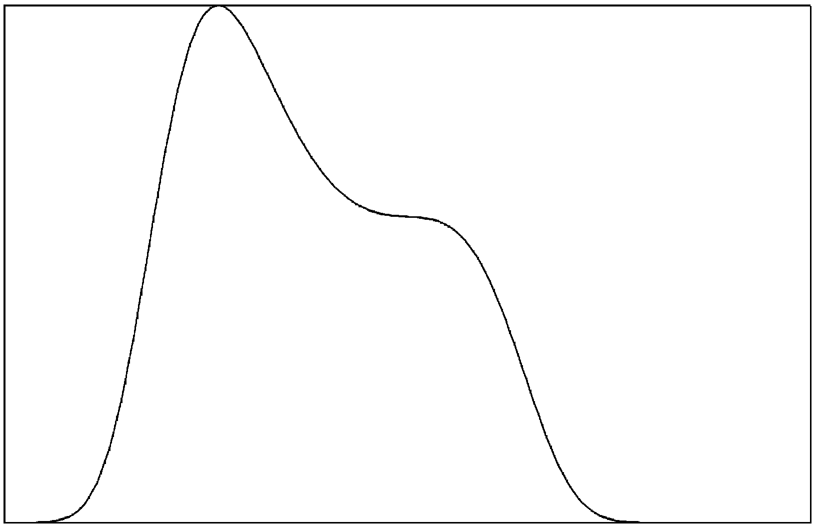


Figure 4.13: Optimal filter density at 9.54.

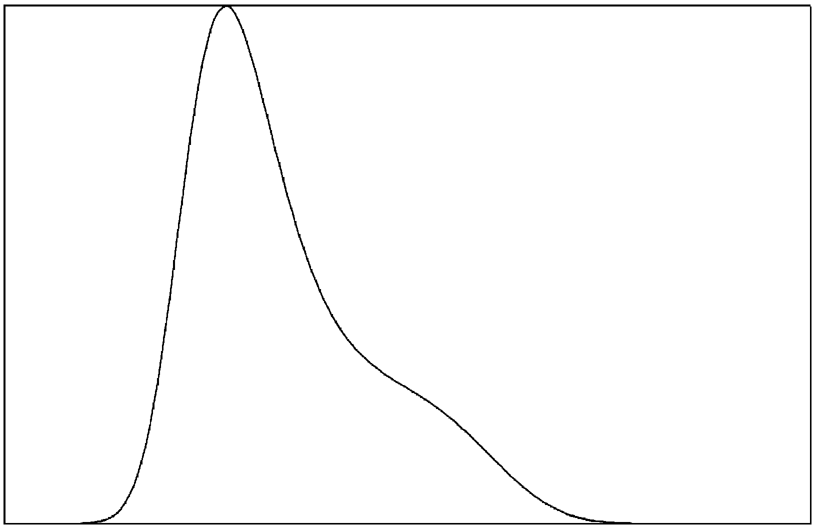


Figure 4.14: Projection filter density at 9.54.

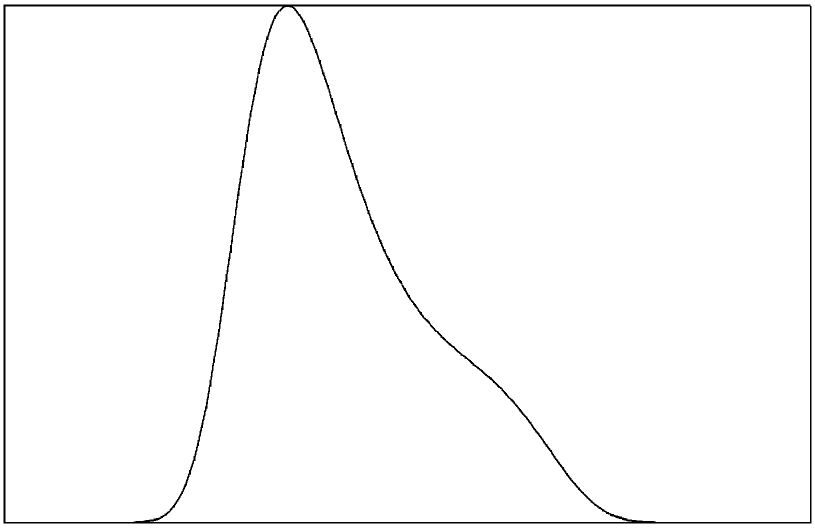


Figure 4.15: Mean projection filter and mean optimal filter between 0 and 5.

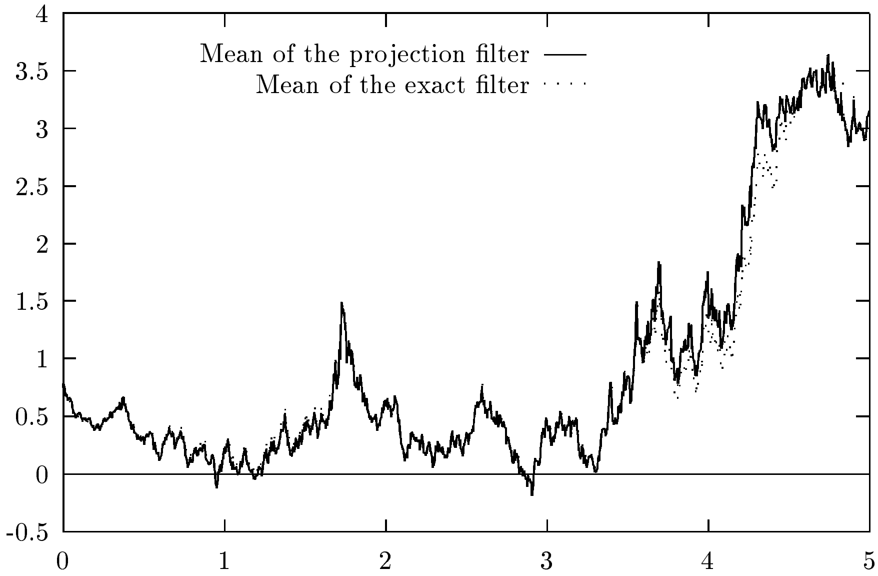


Figure 4.16: Projection residual between 0 and 5.

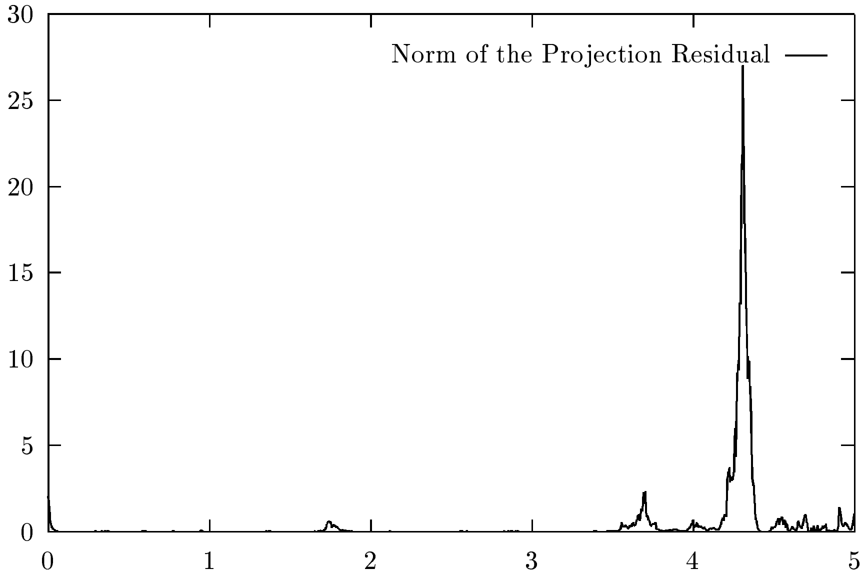


Figure 4.17: Mean projection filter and mean optimal filter between 5 and 10.

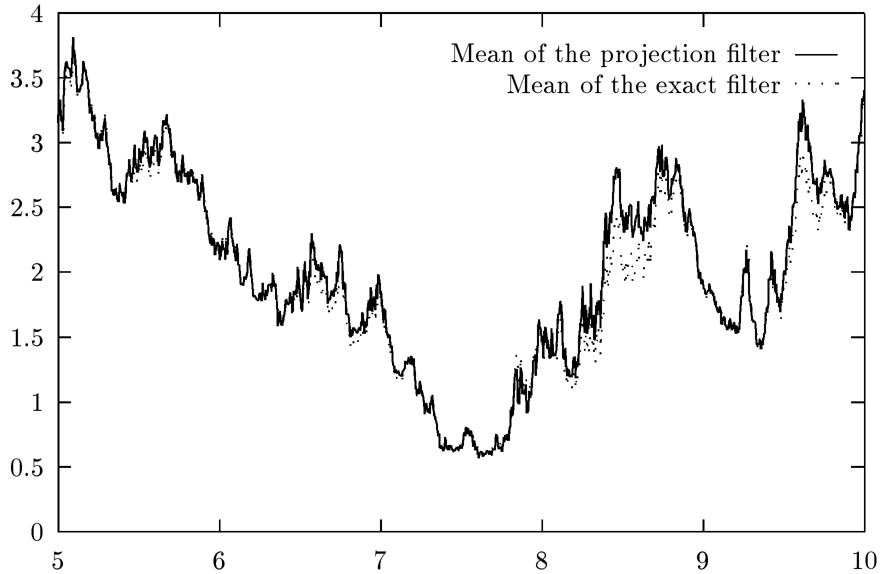


Figure 4.18: Projection residual between 5 and 10.

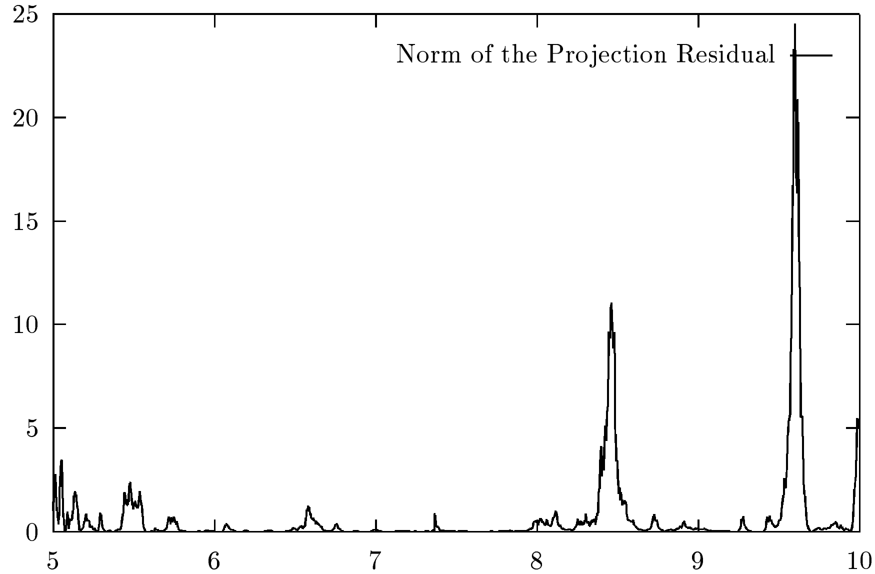


Figure 4.19: True state and mean from the projection filter between 0 and 5.

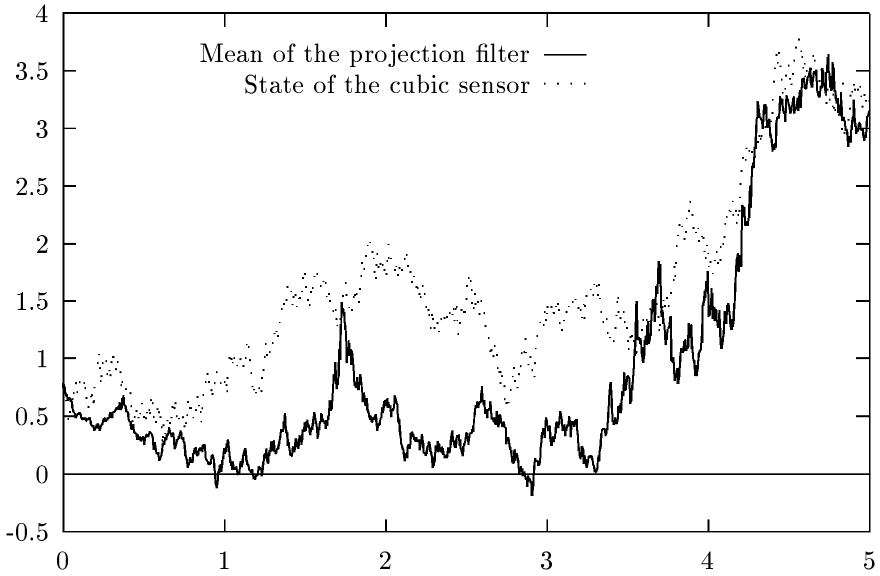


Figure 4.20: True state and mean from the projection filter between 5 and 10.

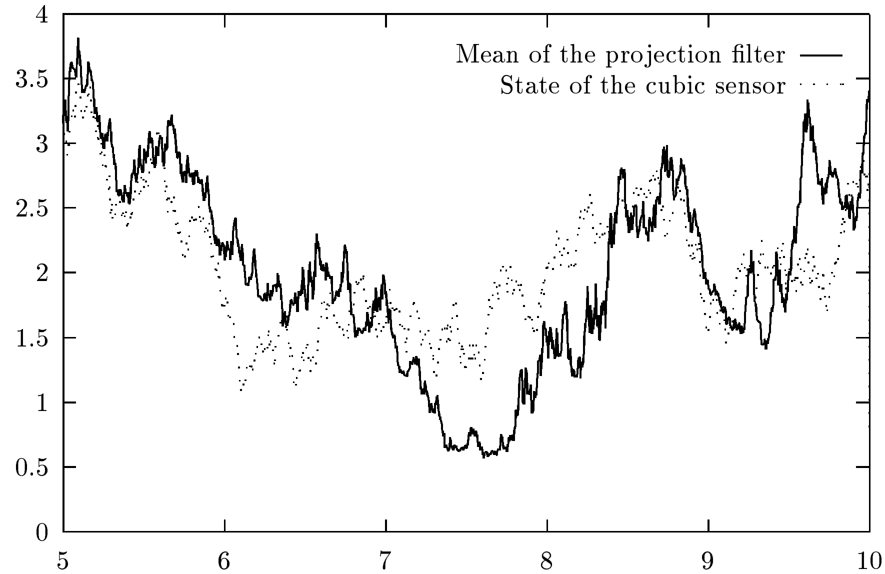


Figure 4.21: True state and mean from the optimal filter between 0 and 5.

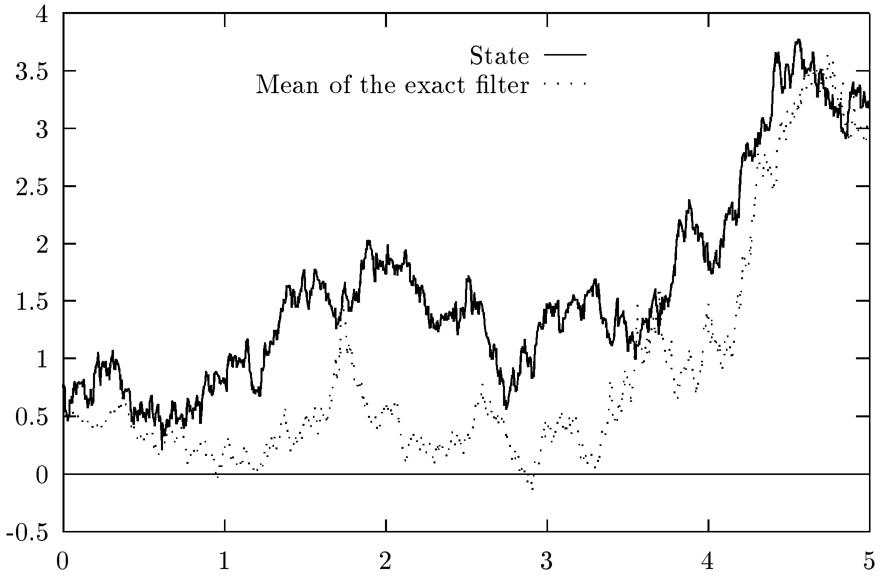
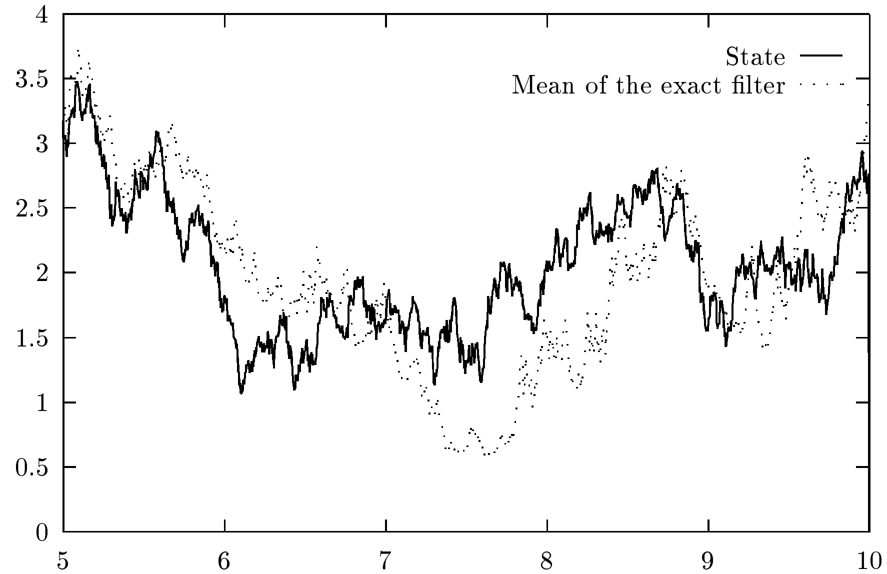


Figure 4.22: True state and mean from the optimal filter between 5 and 10.



4.8 Conclusion

In this Chapter we have introduced a new and systematic way of designing approximate finite–dimensional filters.

One major issue left is the choice of the exponential family S. A first answer has been given in Section 4.5, but this does not completely solve the problem : with the choice of the family $S_\bullet$ there is still some freedom left in the choice of the dimension m and in the choice of the remaining functions $\{c_{s+1}, \cdots, c_m\}$, which could be used to reduce the total residual norm $r_t^* = r_t^\bullet$.

This freedom could also be used to design an adaptive scheme for the choice of the exponential family S.

It would also be useful to obtain for all $t \geq 0$ an estimate of the distance between the optimal–filter density p_t and the projection–filter density p_t^π, in terms of the total residual norm history $\{r_s^*, 0 \leq s \leq t\}$.

Finally, we would like to define projection filters for discrete–time systems, and relate this problem with the work of Kulhavý [40], [41]. Another motivation for this study will be to obtain efficient numerical schemes for the solution of the stochastic differential equation satisfied by the projection–filter parameters, i.e. equation (4.12) for a general family S, or equation (4.20) for the family $S_\bullet$.

Each of these problems requires further investigation, and we hope to address all of them in a subsequent work.

Chapter 5

Assumed Density Filters and Projection Filters

If with a pure mind a person speaks or acts, happiness follows him even as his never–departing shadow

Dhammapada, I.2

Blessed are the pure in heart, for they will see God

Matthew, V.8

5.1 Introduction

In the present chapter we shall see how two completely–different approaches to approximate filtering lead to the same result when dealing with exponential families.

The story so far: As we saw in Chapter 2, the filtering problem consists of estimating the state of a stochastic differential system from noisy observations. In the linear Gaussian case the problem is solved by the well–known Kalman filter, a finite–dimensional system of equations for the first two conditional moments of the state given the observations. As previously remarked, in the linear context this system of equations provides also the complete conditional density of the state given the observations, since this density is Gaussian and hence characterized by the first two moments. In the general nonlinear case, the filtering problem consists of computing the conditional density of the state given the observations. This density is the solution of a stochastic partial differential equation, the Kushner–Stratonovich equation, which was introduced in Section 2.2. The general nonlinear problem is far more complicated because the resulting nonlinear filter is not finite dimensional in general.

An approximation method in the nonlinear case is the assumed–density filter (ADF). The ADF is obtained from the selection of a few moment equations, which are closed under the assumption that the density is of a certain form, e.g. Gaussian, etc. We present a detailed definition of the assumed–density filters in Section 5.2. However, the ADF is an approximation and as such has advantages and disadvantages. It is illogical to assume that the conditional density is Gaussian while in general it is not Gaussian. This logical inconsistency manifests itself when one compares the assumed–density filter obtained by using Itô calculus with the assumed–density filter obtained if McShane-Fisk-Stratonovich (MFS) calculus is used instead. We present an example which shows that the MFS–based ADF and the Itô–based ADF are not directly related by Itô–MFS transformations, i.e. the MFS–based ADF is not just an MFS version of the Itô–based ADF.

In Chapter 4 we introduced the projection filter (PF), which is a finite–dimensional nonlinear filter based on the differential–geometric approach to statistics. We also considered the projection filter particularized to exponential families in the framework of SDE's on manifolds. The PF is obtained by orthogonally projecting the right–hand side of the Kushner–Stratonovich equation onto the tangent space of a finite–dimensional manifold of probability densities, according to the Fisher metric and its extension to infinite–dimensional space of square roots of densities, known as the Hellinger distance. In 1991, Hanzon and Hut have proved formally in [29] that if one projects orthogonally onto the tangent space of the finite–dimensional manifold of Gaussian densities, the resulting PF coincides with the MFS–based Gaussian assumed–density filter. The performance of this filter will be studied in the case of small observation noise in Chapter 6, and is based on the results given in Brigo [6] and [8].

In the present chapter we intend to present a full proof of the abovementioned equivalence. In fact a much more general result will be shown, namely that the PF coincides with an MFS–based ADF for any exponential family. As a consequence the projection filter for exponential families can be obtained as an MFS–based ADF, and the filter formulas can be obtained easily from the moment equations. At the same time this equivalence yields a remedy to the lack of logical consistency involved in the definition of the assumed density filters : the MFS–based ADF that updates the moment parameters of an exponential distribution is a well–defined concept, because of its interpretation as a projection filter.

A short description of this chapter is as follows : The assumed–density filter is introduced in Section 5.2. We conclude by proving the equivalence between ADF and PF for exponential families in Section 5.3, where we also present an example to show that this equivalence does not hold for general (non–exponential) families. Part of the material of this chapter has already appeared in [16].

5.2 Assumed density filters

Because the equations of nonlinear filtering are generally intractable, many approximation methods have been proposed. A well-known approximation method is the EKF (extended Kalman filter), in which the conditional first and second–order moments are approximated by using a linearization procedure. A potential disadvantage of such a method is that no use is made of the general nonlinear–filtering equations : after linearization the formulas for linear Gaussian filtering are applied. If one tries to develop approximation schemes starting from the nonlinear–filtering equations, one is confronted with the problem that the conditional densities (if they exist) do not belong in general to any finite–dimensional class of densities. One heuristic way to deal with this problem is to consider the moment equations and to assume arbitrarily that the conditional densities belong to some finite–dimensional class of densities, even if this is known to be wrong. The resulting moment equations will in general be *inconsistent*, but by selecting carefully a limited number of moment equations one can obtain a consistent definition of an approximate filter, which is called an assumed–density filter in the literature, see Kushner [42], and Maybeck [47, Section 12.7].

As will be shown, it also matters whether the selected moment equations are taken in Itô or in MFS form. In order to discuss such assumed–density filters properly, and to prove the relation with the projection filters in Section 5.3 below, we give now a more formal definition of assumed–density filters.

Consider a function $c : \mathbf{R}^n \to \mathbf{R}^m$. The following set of assumptions will be in force throughout the chapter:

The function c is twice differentiable and, together with its derivatives up to order 2, has at most polynomial growth when $|x|$ goes to infinity. Assume that, in addition to assumptions (A), (B) and (C) of Section 2.2, the coefficients f_t and a_t of the system (2.1) have at most polynomial growth when $|x|$ goes to infinity.

Then the conditions given in [27] are fulfilled for the c–moments to satisfy (2.3), i.e.

$$\begin{aligned} d\,\pi_t(c) &= \pi_t(\mathcal{L}_t\,c)\,dt - \tfrac{1}{2}\,[\pi_t(|h_t|^2\,c) - \pi_t(|h_t|^2)\,\pi_t(c)]\,dt \\ &\quad + \sum_{k=1}^{d}[\pi_t(h_t^k\,c) - \pi_t(h_t^k)\,\pi_t(c)] \circ dY_t^k\ . \end{aligned} \tag{5.1}$$

The Itô version of this equation is obtained from (2.2) by setting $\phi = c$, and holds under the conditions just described.

The following is a generalization of the concept of assumed conditional–probability density filters as introduced in [42].

Definition 5.2.1 *Consider a finite set* $\{c_1, \cdots, c_m\}$ *of twice–differentiable scalar functions defined on* $\mathbf{R}^n$, $c : \mathbf{R}^n \to \mathbf{R}^m$, *such that each* c_i, $i = 1, \cdots, m$ *and its derivatives up to order 2 have at most polynomial growth. Consider a corresponding* m*–dimensional family* $\{\pi(\cdot, \eta), \eta = (\eta_1, \cdots, \eta_m) \in \mathcal{E}\}$ *of probability measures, where* $\mathcal{E} \subset \mathbf{R}^m$ *is open, such that each element of the family satisfies the equations*

$$\eta_i = E_\eta\{c_i\}\ , \qquad i = 1, \cdots, m$$

and is uniquely *specified by these equations. Here* $E_\eta\{\cdot\}$ *denotes the expectation with respect to the probability measure* $\pi(\cdot, \eta)$.

In accordance with the ADF principle, the Itô–based ADF is defined by the Itô stochastic differential equations

$$d\eta_t^i = E_{\eta_t}\{\mathcal{L}_t\,c_i\}\,dt + \sum_{k=1}^{d}[\,E_{\eta_t}\{h_t^k\,c_i\} - E_{\eta_t}\{h_t^k\}\,\eta_t^i\,]\,[\,dY_t^k - E_{\eta_t}\{h_t^k\}\,dt\,]\ , \tag{5.2}$$

for $i = 1, \cdots, m$. *Similarly the MFS–based ADF is defined by the MFS stochastic differential equations*

$$\begin{aligned} d\eta_t^i &= E_{\eta_t}\{\mathcal{L}_t\,c_i\}\,dt - \tfrac{1}{2}\,[\,E_{\eta_t}\{|h_t|^2\,c_i\} - E_{\eta_t}\{|h_t|^2\}\,\eta_t^i\,]\,dt \\ &\quad + \sum_{k=1}^{d}[\,E_{\eta_t}\{h_t^k\,c_i\} - E_{\eta_t}\{h_t^k\}\,\eta_t^i\,] \circ dY_t^k\ , \qquad i = 1, \cdots, m. \end{aligned} \tag{5.3}$$

Although this may be surprising at first, the Itô–based ADF and the MFS–based ADF are *different* filters in general. This will be shown by working out the Itô–based and MFS–based Gaussian assumed–density filters for the cubic sensor problem. The fact that they are different is due to the *inconsistency* that is inherent to the ADF–concept : selecting a different set of equations to which it is applied leads to different results.

Example 5.2.2 (MFS–based Gaussian ADF for the cubic sensor.) *We consider the scalar system*

$$dX_t = \sigma\, dW_t \ ,$$

$$dY_t = X_t^3\, dt + dV_t \ ,$$

with the usual independence assumptions for the standard Brownian motions $\{W_t\,,\,t \geq 0\}$ *and* $\{V_t\,,\,t \geq 0\}$*, and where* σ *is a real constant. Let us compute the MFS–based ADF for this system using a Gaussian family , i.e. choosing* $c_1(x) = x$*, and* $c_2(x) = x^2$*. Then one obtains* $\mu = \eta_1 = E_\eta\{x\}$*, and* $\eta_2 = E_\eta\{x^2\}$*, which indeed parametrize the Gaussian family over* $\mathbf{R}$*. Define* $P := E_\eta\{(x-\mu)^2\} = \eta_2 - \eta_1^2$*. In the Gaussian case one has the following relations between the centered higher order moments up to order six, and the variance* P

$$E_\eta\{x-\mu\} = E_\eta\{(x-\mu)^3\} = E_\eta\{(x-\mu)^5\} = 0 \ ,$$

$$E_\eta\{(x-\mu)^2\} = P \ , \qquad E_\eta\{(x-\mu)^4\} = 3\,P^2 \ , \qquad E_\eta\{(x-\mu)^6\} = 15\,P^3 \ . \tag{5.4}$$

Making use of relations (5.4), equation (5.3) results in the following MFS–based Gaussian ADF :

$$\begin{aligned} d\mu_t &= (-3\,\mu_t^5\,P_t - 30\,\mu_t^3\,P_t^2 - 45\,\mu_t\,P_t^3)\,dt + (3\,\mu_t^2\,P_t + 3\,P_t^2) \circ dY_t \ , \\ dP_t &= (\sigma^2 - 15\,\mu_t^4\,P_t^2 - 90\,\mu_t^2\,P_t^3 - 45\,P_t^4)\,dt + 6\,\mu_t\,P_t^2 \circ dY_t \ . \end{aligned} \tag{5.5}$$

This should (and will) be compared with the Itô–based ADF for the same problem, with the same family of probability densities and the same choice of functions c_1 and c_2.

Example 5.2.3 (Itô–based Gaussian ADF for the cubic sensor.) *Using relations (5.4), equation (5.2) results in the following Itô–based Gaussian ADF :*

$$\begin{aligned} d\mu_t &= (-3\,\mu_t^5\,P_t - 12\,\mu_t^3\,P_t^2 - 9\,\mu_t\,P_t^3)\,dt + (3\,\mu_t^2\,P_t + 3\,P_t^2)\,dY_t \ , \\ dP_t &= (\sigma^2 - 15\,\mu_t^4\,P_t^2 - 36\,\mu_t^2\,P_t^3 - 9\,P_t^4)\,dt + 6\,\mu_t\,P_t^2\,dY_t \ . \end{aligned} \tag{5.6}$$

Putting these Itô equations in MFS form *one obtains the MFS version of the Itô–based ADF :*

$$\begin{aligned} d\mu_t &= (-3\,\mu_t^5\,P_t - 30\,\mu_t^3\,P_t^2 - 36\,\mu_t\,P_t^3)\,dt + (3\,\mu_t^2\,P_t + 3\,P_t^2) \circ dY_t \ , \\ dP_t &= (\sigma^2 - 15\,\mu_t^4\,P_t^2 - 81\,\mu_t^2\,P_t^3 - 18\,P_t^4)\,dt + 6\,\mu_t\,P_t^2 \circ dY_t \ . \end{aligned} \tag{5.7}$$

By comparing the MFS–based Gaussian ADF given in (5.5) with the MFS version of the Itô–based Gaussian ADF given in (5.7), we see that these two filters are different, because their representations as MFS stochastic differential equations differ.

As is clear from the definition, the construction of an ADF involves both the choice of functions $\{c_1, \cdots, c_m\}$ and the choice of an m–dimensional family of probability distributions which are characterized uniquely by the vector $\eta = (\eta_1, \cdots, \eta_m)$, where $\eta_i = E_\eta\{c_i\}$ for $i = 1, \cdots, m$, and also the choice of a stochastic calculus, either Itô or MFS. Suppose that one wants to work with a number of functions $\{c_1, \cdots, c_m\}$ and their corresponding expectation parameters $\{\eta_1, \cdots, \eta_m\}$. Then one way to obtain a family of densities which has the desired property is by using the concept of maximum entropy : given $\{c_1, \cdots, c_m\}$ and their expectation parameters $\{\eta_1, \cdots, \eta_m\}$, choose the probability density p with maximal entropy under the conditions $E_p\{c_i\} = \eta_i$ for all $i = 1, \cdots, m$. This is possible if η is chosen such that there exists at least one probability density whose moments have these values. The solution to this problem, see Kagan, Linnik and Rao [33], is given by the exponential family $\{\exp[\theta^T\, c(x) - \psi(\theta)]\,,\, \theta \in \Theta\}$, which was presented in Section 3.3. In the following sections it will be shown that if such an exponential family is chosen, then the MFS–based ADF can be interpreted as a projection filter. The projection filters are consistently defined and therefore do not lead to the inconsistencies which were found for the ADF's.

5.3 Equivalence between ADF and PF

The main theorem of the chapter can now be stated, for which we shall present two different proofs. The first proof is more elegant and concise, but it does not give much insight in the geometric nature of the result. The second proof will rely more on geometric concepts. It will make explicit use of projections on the tangent spaces, and will rely on a crucial result from the theory of *information geometry*, i.e. the biorthogonality relations between the tangent vectors corresponding to the canonical parameters, and the tangent vectors corresponding to the expectation parameters, see [2]. The initial condition is assumed to be the same $p_0 = p(\cdot, \theta_0)$ *in* S for both filters.

Theorem 5.3.1 *For any exponential family S , the projection filter coincides with the MFS–based assumed density filter.*

First Proof. We start from equation (4.12) for the projection filter canonical parameters, i.e.

$$g(\theta_t) \circ d\theta_t = E_{p(\cdot,\theta_t)}\{\mathcal{L}_t\, c\}\, dt - E_{p(\cdot,\theta_t)}\{\tfrac{1}{2}\, |h_t|^2\, [c - \eta(\theta_t)]\}\, dt$$
$$+ \sum_{k=1}^{d} E_{p(\cdot,\theta_t)}\{h_t^k\, [c - \eta(\theta_t)]\} \circ dY_t^k \ .$$

According to Remark 3.3.4, the expectation parameters can be expressed in terms of the canonical parameters as

$$\eta_i = \eta_i(\theta) = E_{p(\cdot,\theta)}\{c_i\} = E_{p_E(\cdot,\eta)}\{c_i\} \ ,$$

with derivatives

$$\frac{\partial}{\partial\theta_j}\eta_j(\theta) = g_{ij}(\theta) \ .$$

The chain rule for MFS integrals immediately gives

$$d\eta_t = g(\theta_t) \circ d\theta_t = E_{p_E(\cdot,\eta_t)}\{\mathcal{L}_t\, c\}\, dt - E_{p_E(\cdot,\eta_t)}\{\tfrac{1}{2}\, |h_t|^2\, [c - \eta_t]\}\, dt$$
$$+ \sum_{k=1}^{d} E_{p_E(\cdot,\eta_t)}\{h_t^k\, [c - \eta_t]\} \circ dY_t^k \ ,$$

which is exactly equation (5.3) obtained using the assumed density filter idea. □

For the second proof of Theorem 5.3.1, we shall need the following result.

Theorem 5.3.2 *Let $S^{1/2}$ and $\Sigma^{1/2}_{t_0,\theta_0}$ be respectively the manifold associated with the exponential family (4.5), and its enveloping manifold as defined in (4.7) and (4.8). For simplicity, we use the notations $p_0(\cdot,\theta,\xi) = p_{t_0,\theta_0}(\cdot,\theta,\xi)$, and $\psi_0(\theta,\xi) = \psi_{t_0,\theta_0}(\theta,\xi)$. Define the following expectation parameters*

$$\eta_i(\theta) := \frac{\partial}{\partial\theta_i}\psi(\theta) = E_{p(\cdot,\theta)}\{c_i\} \ , \qquad i = 1,\cdots,m \ ,$$

and

$$\bar\eta_i(\theta,\xi) := \frac{\partial}{\partial\theta_i}\psi_0(\theta,\xi) = E_{p_0(\cdot,\theta,\xi)}\{c_i\} \ , \qquad i = 1,\cdots,m \ ,$$

$$\bar\chi_l(\theta,\xi) := \frac{\partial}{\partial\xi_l}\psi_0(\theta,\xi) = E_{p_0(\cdot,\theta,\xi)}\{d_l - \xi_l\, |\xi|^2\, |d|^4\} \ , \qquad l = 1,\cdots,s \ , \tag{5.8}$$

respectively. Introduce also the following notations for the tangent vectors associated with the different parametrizations, i.e.

$$\partial_i(\theta) := \frac{\partial}{\partial\theta_i}\sqrt{p(\cdot,\theta)} \ , \qquad \partial^i(\theta) := \frac{\partial}{\partial\eta_i}\sqrt{p(\cdot,\theta)} \ , \qquad i = 1,\cdots,m \ ,$$

and

$$\partial_i(\theta,\xi) := \frac{\partial}{\partial\theta_i}\sqrt{p_0(\cdot,\theta,\xi)}\ , \qquad \partial^i(\theta,\xi) := \frac{\partial}{\partial\bar{\eta}_i}\sqrt{p_0(\cdot,\theta,\xi)}\ ,$$

$$\partial_{m+l}(\theta,\xi) := \frac{\partial}{\partial\xi_l}\sqrt{p_0(\cdot,\theta,\xi)}\ , \qquad \partial^{m+l}(\theta,\xi) := \frac{\partial}{\partial\bar{\chi}_l}\sqrt{p_0(\cdot,\theta,\xi)}\ ,$$

for $i=1,\cdots,m$ and $l=1,\cdots,s$, respectively.

Given any tangent vector w to the manifold $\Sigma^{1/2}_{t_0,\theta_0}$ at point $\sqrt{p(\cdot,\theta)}=\sqrt{p_0(\cdot,\theta,0)}$, which we decompose on the basis associated with the expectation parameters, as

$$w = \sum_{i=1}^{m} w_i\,\partial^i(\theta,0) + \sum_{l=1}^{s} w_{m+l}\,\partial^{m+l}(\theta,0)\ ,$$

the projection of w onto the tangent space of $S^{1/2}$ at $\sqrt{p(\cdot,\theta)}$ satisfies

$$\Pi_{t_0,\theta_0}\,w = \sum_{i=1}^{m} w_i\,\partial^i(\theta)\ .$$

Proof : We first prove that the expectation parameters $\bar{\eta}=(\bar{\eta}_1,\cdots,\bar{\eta}_m)$ and $\bar{\chi}=(\bar{\chi}_1,\cdots,\bar{\chi}_s)$ provide another parametrization of the enveloping manifold, i.e. we prove that the Jacobian matrix $\bar{J}(\theta,\xi)$ of the transformation $(\theta,\xi)\mapsto(\bar{\eta},\bar{\chi})$ is invertible.

From Proposition 4.4.1 above, the Fisher information matrix $\bar{g}(\theta,\xi)$ of the enveloping manifold is invertible for any $\theta\in\Theta$, and any $\xi\in\Xi$. It follows by easy calculations that

$$\bar{J}(\theta,\xi) = \bar{g}(\theta,\xi) - E_{p_0(\theta,\xi)}\{|d|^4\}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & R(\xi)\end{array}\right]\ ,$$

where the $s\times s$ matrix $R(\xi)$ is given by

$$R(\xi) = |\xi|^2\,I + 2\,\xi\,\xi^T,$$

for any $\xi\in\Xi$. By the Lebesgue dominated convergence theorem

$$\lim_{\xi\to 0}|\xi|^2\ E_{p_0(\theta,\xi)}\{|d|^4\}$$

$$= \lim_{\xi\to 0} E_{p(\cdot,\theta)}\{|\xi|^2\ |d|^4\ \exp[\xi^T d - \tfrac{1}{4}\,|\xi|^4\,|d|^4]\}\ \exp[\psi(\theta)-\psi_0(\theta,\xi)]\ = 0\ .$$

Therefore, the Jacobian matrix $\bar{J}(\theta,\xi)$ is invertible, provided ξ is sufficiently small, and in addition

$$\bar{g}(\theta,\xi)\ [\bar{J}(\theta,\xi)]^{-1} = \left[\begin{array}{c|c} I & 0 \\ \hline * & *\end{array}\right]\ ,$$

where the asterisks indicate entries that need not be specified here.

It results from this observation, that the following partial *biorthogonality* relations hold

$$\begin{aligned} \langle \partial_j(\theta,\xi), \partial^i(\theta,\xi) \rangle &= \tfrac{1}{4}\,\delta_{i,j}\ , & i &= 1,\cdots,m \\ \langle \partial_j(\theta,\xi), \partial^{m+l}(\theta,\xi) \rangle &= 0\ , & l &= 1,\cdots,s \end{aligned} \tag{5.9}$$

for all $j = 1,\cdots,m$.

Finally, it is easily checked that for all $\theta \in \Theta$

$$\partial_j(\theta) = \partial_j(\theta,0)\ ,$$

for all $j = 1,\cdots,m$. We notice that by definition $(w - \Pi_{t,\theta}\, w)$ is orthogonal to the tangent space of $S^{1/2}$ at $\sqrt{p(\cdot,\theta)}$, hence

$$\langle \partial_j(\theta), w - \Pi_{t_0,\theta_0}\, w \rangle = 0\ ,$$

for all $j = 1,\cdots,m$. Therefore

$$\langle \partial_j(\theta), \Pi_{t_0,\theta_0}\, w \rangle = \langle \partial_j(\theta,0), w \rangle$$

$$= \sum_{i=1}^{m} w_i\, \langle \partial_j(\theta,0), \partial^i(\theta,0) \rangle + \sum_{l=1}^{s} w_{m+l}\, \langle \partial_j(\theta,0), \partial^{m+l}(\theta,0) \rangle = \tfrac{1}{4}\, w_j\ ,$$

because of the biorthogonality relations (5.9), hence the projected vector is of the announced form. □

Now we can state the more geometrical proof of Theorem 5.3.1.

SECOND PROOF OF THEOREM 5.3.1. According to Theorem 4.4.3, the exponential projection filter equation is obtained by projecting the tangent vectors of $\Sigma^{1/2}_{t,\theta_t}$ which appear in the right–hand side of the Kushner–Stratonovich equation (2.7) onto the tangent space of $S^{1/2}$. If we decompose these tangent vectors on the basis associated with the expectation parameters, we obtain

$$\begin{aligned} \mathcal{P}_t(\sqrt{p_E(\cdot,\eta_t)}) &= \sum_{i=1}^{m} p_i(\eta_t)\, \partial^i(\theta_t,0) + \sum_{l=1}^{s} p_{m+l}(\eta_t)\, \partial^{m+l}(\theta_t,0) \\ \mathcal{Q}^k_t(\sqrt{p_E(\cdot,\eta_t)}) &= \sum_{i=1}^{m} q^k_i(\eta_t)\, \partial^i(\theta_t,0) + \sum_{l=1}^{s} q^k_{m+l}(\eta_t)\, \partial^{m+l}(\theta_t,0)\ , \end{aligned}$$

for $k = 0,1,\cdots,d$. Notice that, from the biorthogonality relations (5.9) and the expression for $\mathcal{P}_t\ (\sqrt{p_E(\cdot,\eta_t)})$ presented in Section 4.4 it holds

$$p_i(\eta_t) \quad = \quad 4\,\langle \mathcal{P}_t(\sqrt{p_E(\cdot,\eta_t)}), \partial_i(\theta_t) \rangle$$

$$= \quad 4\,\langle \tfrac{1}{2}\,\sqrt{p_E(\cdot,\eta_t)}\,\alpha_{t,\theta_t}\,,\,\tfrac{1}{2}\,\sqrt{p_E(\cdot,\eta_t)}\,[c_i - \eta_t^i]\rangle$$

$$= \quad E_{p_E(\cdot,\eta_t)}\{\alpha_{t,\theta_t}\,[c_i - \eta_t^i]\}\ ,$$

and similarly

$$q_i^k(\eta_t) = E_{p_E(\cdot,\eta_t)}\{\beta_{t,\theta_t}^k\,[c_i - \eta_t^i]\}\ ,$$

for $k = 0, 1, \cdots, d$. The projections of these tangent vectors determine the right–hand side of the stochastic differential equation for the projection filter. According to Theorem 5.3.2, such projections read

$$\begin{aligned} \Pi_{t,\theta_t} \circ \mathcal{P}_t(\sqrt{p_E(\cdot,\eta_t)}) &= \sum_{i=1}^m p_i(\eta_t)\,\partial^i(\theta_t) \\ \Pi_{t,\theta_t} \circ \mathcal{Q}_t^k(\sqrt{p_E(\cdot,\eta_t)}) &= \sum_{i=1}^m q_i^k(\eta_t)\,\partial^i(\theta_t)\ , \end{aligned}$$

for $k = 0, 1, \cdots, d$. Expanding $d\sqrt{p_E(\cdot,\eta_t)}$ with the chain rule and collecting tangent vectors on both sides, yields the following stochastic differential equation for the projection filter

$$d\eta_t^i = p_i(\eta_t)\,dt - q_i^0(\eta_t)\,dt + \sum_{k=1}^d q_i^k(\eta_t) \circ dY_t^k\ , \qquad i = 1, 2, \cdots, m\ ,$$

which concludes the proof. □

The equivalence between the MFS–based ADF and the PF is shown to hold for exponential families. In general, for other families of distributions such equivalence does *not* hold. This can be seen from the following simple example in which we consider a particular *curved* (Gaussian) exponential family.

Example 5.3.3 (Projection filter with a curved Gaussian family.) *We consider the scalar system*

$$\begin{aligned} dX_t &= f(X_t)\,dt + \sigma(X_t)\,dW_t \\ dY_t &= X_t\,dt + dV_t\ , \end{aligned}$$

where the coefficients are supposed to satisfy the usual assumptions. Choose the following curved family of Gaussian densities :

$$\{p(x,\theta) = \exp[\theta\,x - \theta^2\,x^2 - \psi(\theta)]\,,\,\theta \in \mathbf{R} \setminus \{0\}\}\ ,$$

where $p(\cdot,\theta)$ is the Gaussian density with mean $1/(2\theta)$, and variance $1/(2\theta^2)$. We shall denote by $E_\theta\{\cdot\}$ the expectation w.r.t. the density $p(\cdot,\theta)$. Notice that $\eta = E_\theta\{x\} = 1/(2\theta)$. The densities in the above curved Gaussian family may be characterized by η as well. We denote by $E_\eta\{\cdot\}$ the corresponding expectation, so that for example

$$E_\eta\{\ f\} = \int_{-\infty}^{+\infty} f(x) \frac{1}{\sqrt{2\pi 2\eta^2}} \exp[-\tfrac{1}{2}\frac{(x-\eta)^2}{2\eta^2}]\ d\lambda(x).$$

Consider the general equation (4.3) for the projection filter, and notice that, since $\eta = 1/(2\theta)$, we have $d\eta_t = -1/(2\theta_t^2) \circ d\theta_t$. This results in the following projection filter :

$$d\eta_t = -\frac{1}{5}\,[\,E_{\eta_t}\{f\} - \frac{2}{\eta_t}\,E_{\eta_t}\{x\,f\} - \frac{2}{\eta_t}\,E_{\eta_t}\{\sigma\} + 6\,\eta_t^3\,]\,dt + \frac{2}{5}\,\eta_t^2 \circ dY_t\ .$$

On the other hand, equation (5.3) yields instead

$$d\eta_t = [\,E_{\eta_t}\{f\} - \frac{5}{2}\,\eta_t^3\,]\,dt + 2\,\eta_t^2 \circ dY_t\ ,$$

making use of relations (5.4).

Anyway, one of the striking features of the Theorem 5.3.1 is that it yields a characterization of the projection filters for exponential families in terms of assumed density filters, which are not intrinsically based on differential geometry, and can be understood without using geometric concepts.

Finally we observe that as the Itô–based and MFS–based ADF are different, the theorems proved above state that for a general exponential family S the equivalence with the projection filter holds only for the MFS–based ADF. However, it can be shown that the MFS–based and the Itô–based ADF coincide for special choices of the exponential family, such as the families $S_\bullet$ and S_* introduced in Section 4.5 which are constructed in such a way that the observation functions h^k for $k = 1, \cdots, d$ are contained in the linear space $\text{span}\{c_1, \cdots, c_m\}$. Indeed, the following theorem holds:

Theorem 5.3.4 *For the exponential family S_*, the Itô–based assumed density filter coincides with the MFS–based assumed density filter.*

Proof: It follows from (4.19) that

$$\tfrac{1}{2}\,|h|^2 = \tfrac{1}{2}\sum_{k=1}^{d} |h^k|^2 = \tfrac{1}{2}\sum_{k=1}^{d}\sum_{l,l'=1}^{s} \lambda_l^k\,\lambda_{l'}^k\,c_l\,c_{l'}\ .$$

By specializing to the exponential family S_* the general equation (5.3) for the MFS–based ADF, and using Lemma 3.3.3, we obtain

$$
\begin{aligned}
d\eta_t^i &= E_{\eta_t}\{\mathcal{L}_t\, c_i\}\, dt - \tfrac{1}{2} \sum_{k=1}^{d} \sum_{l,l'=1}^{s} \lambda_l^k\, \lambda_{l'}^k\; [\, E_{\eta_t}\{c_l c_{l'}\, c_i\} - E_{\eta_t}\{c_l c_{l'}\}\, \eta_t^i\,]\, dt \\
&\quad + \sum_{k=1}^{d} \sum_{l=1}^{s} \lambda_l^k\; [\, E_{\eta_t}\{c_l c_i\} - E_{\eta_t}\{c_l\}\, \eta_t^i\,] \circ dY_t^k \\
&= E_{\eta_t}\{\mathcal{L}_t\, c_i\}\, dt - \sum_{k=1}^{d} \sum_{l,l'=1}^{s} g_{il}(\eta_t)\, \lambda_l^k\, \lambda_{l'}^k\, \eta_t^{l'}\, dt \\
&\quad - \tfrac{1}{2} \sum_{k=1}^{d} \sum_{l,l'=1}^{s} \frac{\partial}{\partial \theta_{l'}} g_{il}(\eta_t)\, \lambda_l^k\, \lambda_{l'}^k\, dt + \sum_{k=1}^{d} \sum_{l=1}^{s} g_{il}(\eta_t)\, \lambda_l^k \circ dY_t^k \ ,
\end{aligned}
$$

for $i = 1, \cdots, m$. It is easily checked that the Itô–MFS transformation yields

$$
g_{il}(\eta_t)\, dY_t^k = g_{il}(\eta_t) \circ dY_t^k - \tfrac{1}{2} \sum_{l'=1}^{s} \frac{\partial}{\partial \theta_{l'}} g_{il}(\eta_t)\, \lambda_{l'}^k\, dt \ ,
$$

for all $k = 1, \cdots, d$ and all $i = 1, \cdots, m$. On the other hand, by specializing to the exponential family S_* the general equation (5.2) for the Itô–based ADF, and using Lemma 3.3.3, we obtain directly

$$
\begin{aligned}
d\eta_t^i &= E_{\eta_t}\{\mathcal{L}_t\, c_i\}\, dt \\
&\quad + \sum_{k=1}^{d} \sum_{l=1}^{s} \lambda_l^k\; [\, E_{\eta_t}\{c_l c_i\} - E_{\eta_t}\{c_l\}\, \eta_t^i\,]\; [\, dY_t^k - \sum_{l'=1}^{s} \lambda_{l'}^k\, E_{\eta_t}\{c_{l'}\}\, dt\,] \\
&= E_{\eta_t}\{\mathcal{L}_t\, c_i\}\, dt - \sum_{k=1}^{d} \sum_{l,l'=1}^{s} g_{il}(\eta_t)\, \lambda_l^k\, \lambda_{l'}^k\, \eta_t^{l'}\, dt + \sum_{k=1}^{d} \sum_{l=1}^{s} g_{il}(\eta_t)\, \lambda_l^k\, dY_t^k \ ,
\end{aligned}
$$

for $i = 1, \cdots, m$. □

Chapter 6

Small Observation Noise

O God, I could be bounded in a nut shell and count myself a king of infinite space, were it not that I have bad dreams.

Hamlet, Act II, Scene II

6.1 Introduction

In the present chapter we examine the Gaussian projection filter with small observation noise. In order to maintain the chapter as self contained as possible, we shall repeat some facts that have already appeared in the previous chapters. This little redundance will be helpful for readers interested only in the small noise setting. In fact the present chapter is almost independent of geometric concepts, and as a first reading can be read independently of the rest of the thesis.

The story so far:

We explained in Chapter 2 that the filtering problem consists of estimating the state of a stochastic system from noise–perturbed observations. In the linear Gaussian case the problem is solved by the Kalman filter. In that chapter we noticed how the more–general nonlinear problem is far more complicated because of infinite dimensionality. We remarked that often the extended Kalman filter (EKF) is used in nonlinear problems, even though its use is usually justified on the basis of heuristic considerations, and not much is known about the quality of its performances, except in the case of small observation noise (see [51] and [52]). In Chapter 5 we introduced the assumed–density filter (ADF). The Gaussian ADF (GADF) represents another choice in the nonlinear case, and is obtained by assuming the conditional density to be Gaussian, closing in this way the set of exact equations for the first two moments and producing a finite–dimensional filter (see [47]). The GADF is not too strong from a mathematical point of view, because from a false hypothesis no interesting statement can be obtained. In Chapter 4 we introduced the projection filter (PF). We saw that the PF is a finite–dimensional nonlinear filter based on the differential–geometric approach to statistics. This filter is based on projection of the nonlinear filtering equation onto a finite–dimensional manifold of densities in Fisher metric. As we saw more in general in Chapter 5 for exponential manifolds, if one projects onto the tangent space of the finite–dimensional manifold of Gaussian densities, the resulting Gaussian projection filter (GPF) coincides with an assumed–density filter which is obtained as follows: one computes the first two conditional moments equations *in McShane–Fisk–Stratonovich (MFS) form*, and then assumes the conditional density to be Gaussian, closing in this way the equations for the first two moments. We call this filter MFS-G-ADF. This result is important because it yields a simple characterization of the GPF which is independent of geometric concepts; on the other hand it shows that, despite the logical inconsistency of its definition, the MFS-G-ADF has a rigorous mathematical characterization. In Chapter 5 it was also proven that what we described above is not the same as assuming a Gaussian density in the *Itô* equations for the first two moments. If we do so and afterwards we transform the obtained filter in MFS form, we obtain a different filter: the MFS-G-ADF is not just an MFS version of the Itô-GADF. In other words, the Itô-MFS transformations and the Gaussian–density assumption do not commute.

Now that we have described the filter to be studied in this chapter, we briefly describe the contents of the chapter.

We shall deal with signals modelled by one–dimensional diffusions, in order to keep the chapter more readable and in the spirit of [51]. We shall assume Lipschitz drift and uniformly–bounded diffusion coefficient for the signal and a bijective output function in the observations (plus some more technical assumptions). This set of assumptions on the system is rather common in small noise analysis, as one can see in [51], [52], and [53].

We start by considering a comparison between the signal (true state) and the GPF estimate. We will prove that, under assumptions different from the one needed for the EKF, the GPF provides an estimate for which the mean–square difference from the true state of the system is bounded by the magnitude of the observation noise. This result has been proved for the exact filter in [51], so that here we prove that our filter features a mean–square error the magnitude of which is bounded in the same way as in the case of the optimal filter. The chapter continues by removing the choice of dimension two for the Gaussian manifold. In the initial sections we use a two–dimensional GPF obtained by projecting the optimal filter onto a *two–dimensional* manifold of Gaussian probability densities. Using dimension two for an approximate filter dealing with a one–dimensional diffusion is rather common: both the Extended Kalman Filter and the (classical Itô-based) ADF feature dimension two in this situation (see [47]). For this reason we begin this chapter with a two–dimensional GPF. We introduce later a one–dimensional GPF and show that its mean–square difference from the true state is bounded in the same way as in the case of the two–dimensional GPF. So, under our assumptions on the system, it turns out that the optimal filter can be tracked efficiently even by fixing the variance on the Gaussian manifold: One has just to allow the mean to 'move'. Moreover, the chapter extends some of the results on the comparison true state – GPF estimate to different models. More precisely, if the drift of the system is bounded (and not necessarily Lipschitz) we define a filter which has a nice behaviour and does not depend on the drift of the system.

The chapter is concluded by a comparison between the optimal filter and the GPF. The mean–square distance has in this case a bound proportional to the *square* of the observation noise. Part of the material presented in this chapter is based on the articles [6] and [8].

6.2 The MFS-G-ADF

On the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ let us consider a stochastic process $\{X_t, t \geq 0\}$ of diffusion type, adapted to a filtration $\{\mathcal{F}_t, t \geq 0\}$, and a related measurement process $\{Y_t,\ t \geq 0\}$. Let the dynamic and observation equations be of the following form (cf. [47, 19])

$$\begin{aligned} dX_t &= f(X_t)dt + \sigma(X_t)dW_t \\ dY_t &= h(X_t)dt + \sqrt{R(t)}dV_t. \end{aligned} \tag{6.1}$$

The above are Itô stochastic differential equations. If $\{v_t, t \geq 0\}$ is a Brownian motion, we will write $[...]dv_t$ if we are working with an Itô differential equation, whereas we use the symbol $[...] \circ dv_t$ to specify a McShane-Fisk-Stratonovich (MFS) stochastic differential equation. In (6.1) the symbols have the following meaning: $X_t \in \mathbf{R}$ is the state vector at time t; f and σ are real valued functions, and $\{W_t, t \geq 0\}$ with $W_t \in \mathbf{R}$ a standard Brownian motion process, independent of the initial condition X_0; $Y_t \in \mathbf{R}$ is the stochastic measurement process, h is a real valued function and $\{V_t, t \geq 0\}$ with $V_t \in \mathbf{R}$ a standard Brownian motion independent both of $\{W_t,\ t \geq 0\}$ and of the initial condition

X_0. We assume $R(\cdot)$ to be uniformly–positive: $R(t) > \delta \geq 0$ for all t, according to [45]. $R(t)$ represents the variance parameter of the observation noise. At the moment assume assumptions on f,σ and h given in Chapter 2 to be in force.

Specialize equation (2.3) to our situation (and consider the obvious modifications due to the fact that we keep $R(t)$ not necessarily equal to one):

$$\begin{aligned} d\pi_t(\phi) &= [\,\pi_t(\mathcal{L}\phi) + \tfrac{1}{2}R(t)^{-1}\pi_t(\phi)\pi_t(h^2)]\; dt \\ &+ \; R(t)^{-1}[\pi_t(\phi h) - \pi_t(\phi)\pi_t(h)] \circ dY_t. \end{aligned} \tag{6.2}$$

We use either $\pi_t((\cdot))$ or $\widehat{(\cdot)}$ to denote the conditional expectation given the σ–algebra $\mathcal{Y}_t$ generated by observations up to time t. Then, by choosing $\phi(x) = x$ and $\phi(x) = x^2$ respectively, one derives from (6.2) the (exact) first two conditional moment equations in MFS form. These equations are given by

$$\begin{aligned} d\widehat{x}_t &= [\pi_t(f) - \tfrac{1}{2}R(t)^{-1}\pi_t((X - \widehat{X_t})h^2)]dt \\ &+ \; R(t)^{-1}\pi_t((X - \widehat{X_t})h) \circ dY_t, \end{aligned} \tag{6.3}$$

$$\begin{aligned} dP_t &= [2\pi_t(f(X)(X - \widehat{X_t})) + \pi_t(\sigma^2) \\ &- \; \tfrac{1}{2}R(t)^{-1}\pi_t(h^2[(X - \widehat{X_t})^2 - P_t])]dt \\ &+ \; R(t)^{-1}\pi_t([(X - \widehat{X_t})^2 - P_t]h) \circ dY_t \end{aligned} \tag{6.4}$$

where $\widehat{X_t} = \pi_t(X)$ and $P_t = \pi_t((X - \widehat{X_t})^2)$. Note that (6.3, 6.4) is not a closed set of stochastic differential equations (SDE): the expectation $\pi_t(\cdot)$ in general involves *all* the moments of the conditional density $p_t := p_{X_t|\mathcal{Y}_t}$, not only the first two. According to the assumed density principle seen in Chapter 5, such set of SDE can be closed by *assuming* p_t to be Gaussian. This amounts to performing the following substitutions in (6.3, 6.4):

$$\begin{aligned} \widehat{X_t} &\rightarrow \widetilde{X_t} \\ P_t &\rightarrow \widetilde{P_t} \\ p_t &\rightarrow p_{\mathcal{N}(\widetilde{X}_t,\widetilde{P}_t)}. \end{aligned} \tag{6.5}$$

This leads to the following closed set of SDE:

$$d\widetilde{X_t} = [\widetilde{E}\{f\} - \tfrac{1}{2}R(t)^{-1}\widetilde{E}\{(X - \widetilde{X_t})h^2\}]dt \tag{6.6}$$

$$+ \quad R(t)^{-1}\widetilde{E}\{(X-\widetilde{X_t})h\}\circ dY_t$$

and

$$\begin{aligned} d\widetilde{P}_t &= [2\widetilde{E}\{f(X)(X-\widetilde{X_t})\}+\widetilde{E}\{\sigma^2\} \qquad (6.7)\\ &- \tfrac{1}{2}R(t)^{-1}\widetilde{E}\{h^2[(X-\widetilde{X_t})^2-\widetilde{P}_t]\}]\ dt\\ &+ R(t)^{-1}\widetilde{E}\{[(X-\widetilde{X_t})^2-\widetilde{P}_t]h\}\circ dY_t, \end{aligned}$$

where we denote the expectation w.r.t. the approximated Gaussian density $p_{\mathcal{N}(\widetilde{X}_t,\widetilde{P}_t)}$ either by the symbol $\widetilde{(\cdot)}$ or by $\widetilde{E}\{(\cdot)\}$. Note that $\widetilde{E}\{(\cdot)\}$ depends on $\mathcal{Y}_t$ via $\widetilde{X_t}$ and $\widetilde{P}_t$. Equations (6.6, 6.7) describe the MFS-G-ADF.

6.3 The GPF for a simplified system.

In order to be able to compare the Gaussian projection filter estimate $\widetilde{X_t}$ to the true state X_t of the system we need to simplify our system.

The set of assumptions we require is given by:

(A) We assume f Lipschitz continuous with Lipschitz constant k.

(B1) We assume σ^2 uniformly bounded:

$$0 < b \leq \sigma(x)^2 \leq B \quad \forall x \in \mathbf{R}.$$

(C1) We assume h bijective and C^2, and $|h_x|$ uniformly bounded: there exist two positive constants H_1 and H_2 such that

$$0 < H_1 \leq |h_x(x)| \leq H_2 \ \ \forall x \in \mathbf{R}.$$

Finally we assume h_{xx} Lipschitz continuous.

Under these assumptions one can assume h to be the identity function without loss of generality. Indeed, we can define a new process $\xi_t := h(X_t)$ which is still a diffusion, the equation of which can be obtained via Itô's lemma. This new system reads (h_x and h_{xx} denote the first two derivatives of h)

$$d\xi_t \quad = \quad [h_x(h^{-1}(\xi_t))f(h^{-1}(\xi_t))+\tfrac{1}{2}\sigma(h^{-1}(\xi_t))^2h_{xx}(h^{-1}(\xi_t))]dt$$

$$
\begin{aligned}
& + \quad h_x(h^{-1}(\xi_t))\sigma(h^{-1}(\xi_t))dW_t \\
dY_t \quad & = \quad \xi_t dt + dV_t
\end{aligned}
$$

so that the new function in the observations is the identity. Assumptions (A) and (B1) still hold for this new system if we require (C1) for h. Notice that uniform boundedness of $|h_x|$ is equivalent to Lipschitz continuity of h and h^{-1}, via the mean–value theorem.

From now on we take h to be identity. Moreover, we shall put $R(t) := \epsilon^2$. The quantity $\epsilon \in \mathbf{R}^+$ represents the magnitude of the observation noise V_t. We are interested in studying the filter behaviour for small ϵ.

This simplified system is:

$$
\begin{aligned}
dX_t &= f(X_t)dt + \sigma(X_t)dW_t \\
dY_t &= X_t dt + \epsilon dV_t
\end{aligned}
\tag{6.8}
$$

By specializing equations (6.6, 6.7) to the system (6.8), and remembering the Gaussian moments formulae, we obtain the MFS-G-ADF for the simplified system:

$$
d\widetilde{X_t} = [\widetilde{E}\{f\} - \tfrac{1}{\epsilon^2}\widetilde{P}_t\widetilde{X_t}]dt + \tfrac{1}{\epsilon^2}\widetilde{P}_t \circ dY_t, \quad \widetilde{X_0} = E\{X_0\}, \tag{6.9}
$$

and

$$
d\widetilde{P}_t = [2\widetilde{E}\{f(X)(X - \widetilde{X_t})\} + \widetilde{E}\{\sigma^2\} - \tfrac{1}{\epsilon^2}\widetilde{P}_t^{\,2}]dt, \quad \widetilde{P_0} = P_0. \tag{6.10}
$$

Note that we should put a superscript on the quantities defined above, as they depend on ϵ. A more complete notation in the above equations would be $\mathcal{Y}_t^\epsilon$, $\widetilde{X_t}^\epsilon$, $\widetilde{P}_t^\epsilon$, $\widehat{X_t}^\epsilon$ and P_t^ϵ. We shall henceforth omit such superscript, except in Corollaries 6.5.2, 6.6.2 and Theorem 6.7.1. Notice also that the stochastic differential equation for $\widetilde{P}_t$ contains no noise term.

6.4 Bounds for $\widetilde{P_t}$ of the GPF

In this section we state a lemma which gives some bounds for $\widetilde{P}_t$, proving that this quantity is bounded by a constant proportional to the observation noise ϵ. Note that $\widetilde{P}_t$ is not the true error variance, but just its approximation based on the arbitrary 'Gaussian–density' assumption. In the following treatment, throughout the chapter, we shall frequently use the Bellman–Gronwall inequality without explicitly referring to it. Moreover, the technique used here recalls the work of Wonham [60] on Riccati differential equations.

Lemma 6.4.1 *Assume (A), (B1) and (C1) are satisfied. Let ϵ_0 be a positive real number satisfying $\epsilon_0 < \min(\sqrt{b}/k, 1)$. Then there exist two positive real constants $C_1(b,k), C_2(B,k)$ such that*

$$C_1(b,k)\epsilon \le \widetilde{P}_t \le C_2(B,k)\epsilon \quad \forall t > \epsilon_0, \quad \forall \epsilon < \epsilon_0$$

where k is the Lipschitz constant for f. The two constants are given by

$$\begin{aligned} C_1(b,k) \quad &:= \quad \min\{\sqrt{b} - k\epsilon_0, \\ &\qquad \max[\frac{\widetilde{P}_0}{\epsilon_0}, (\sqrt{b} - k\epsilon_0)[1 - \exp(-2\sqrt{\epsilon_0^2 k^2 + b})]/2] \ \} \\ C_2(B,k) \quad &:= \quad \max\{2k\epsilon_0 + B, \quad \frac{2\sqrt{k^2\epsilon_0^2 + B}}{1 - \exp(-2\sqrt{k^2\epsilon_0^2 + B})}\} \end{aligned}$$

PROOF: We begin by proving

$$\epsilon < \epsilon_0, \quad t > \epsilon_0 \Rightarrow C_1(b,K)\epsilon \le \widetilde{P}_t.$$

Compute

$$\widetilde{E}\{f(X)(X - \widetilde{X_t})\} = \int (x - \widetilde{X}_t) f(x) p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(x) dx$$

$$\ge \int (x - \widetilde{X}_t)(f(\widetilde{X}_t) - k|x - \widetilde{X}_t|) p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(x) dx$$

$$= -k \int (x - \widetilde{X}_t)^2 p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(x) dx = -k\widetilde{P}_t$$

where we have used the Lipschitz condition for f. Also,

$$\widetilde{E}\{\sigma^2\} = \int \sigma(x)^2 p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(x) dx \ge b$$

where we have used assumption (B1) on σ^2. From these two inequalities we deduce, by (6.10), the following one:

$$\frac{d}{dt}\widetilde{P}_t \ge -2k\widetilde{P}_t + b - \frac{1}{\epsilon^2}\widetilde{P}_t^{\,2}. \tag{6.11}$$

Now define w_t to be the solution of the following ordinary differential equation:

$$\dot{w} = -2kw + b - \frac{1}{\epsilon^2}w^2, \quad w_0 = \widetilde{P_0}. \tag{6.12}$$

From (6.11) and (6.12) it follows by standard differential inequality techniques $\widetilde{P}_t \geq w_t$ for all $t \geq 0$. Equation (6.12) is a scalar Riccati differential equation. Let us first solve the associated algebraic equation:

$$-2kw + b - \frac{1}{\epsilon^2} w^2 = 0.$$

The solutions are given by

$$w_{1,2} = \epsilon^2(-k \mp \sqrt{k^2 + \frac{b}{\epsilon^2}}).$$

Notice that w_1 is negative, whereas w_2 is positive. Notice also that $-w_1 > w_2 > 0$. The solution of (6.12) is

$$w = \frac{w_2 + w_1 N \exp\left(-2t\sqrt{k^2 + \frac{b}{\epsilon^2}}\right)}{1 + N \exp\left(-2t\sqrt{k^2 + \frac{b}{\epsilon^2}}\right)} \tag{6.13}$$

where

$$N := \frac{w_2 - w_0}{w_0 - w_1} < 1, \quad |N| < 1.$$

Now we distinguish between two possible cases:

(a) $w_0 \geq w_2$. The solution $w(t)$ is a decreasing function, which asymptotically approaches the value w_2 as $t \to \infty$ (in the particular case $w_0 = w_2$ we have $w_t = w_2 \quad \forall t > 0$). Hence, assuming $\epsilon < \epsilon_0$

$$\widetilde{P}_t \geq w_t \geq w_2 > \epsilon(\sqrt{b} - k\epsilon_0)$$

provided that the last term is positive ($\epsilon_0 \leq \sqrt{b}/k$).

(b) $w_0 < w_2$. The solution $w(t)$ is an increasing function which asymptotically approaches w_2 as $t \to \infty$. We need to examine two different possibilities:

(b') $w_0 > 0$. Then $w(t) \geq w_0 > \epsilon w_0/\epsilon_0 = \epsilon \widetilde{P}_0/\epsilon_0$ provided that $\epsilon < \epsilon_0$, and we are done.

(b") $w_0 = 0$. In this case we obtain $N = -w_2/w_1 > 0$ and from (6.13) we deduce

$$\begin{aligned}
\widetilde{P}_t \geq w(t) &= w_2 \frac{1 - \exp(-2t\sqrt{k^2 + b/\epsilon^2})}{1 + N\exp(-2t\sqrt{k^2 + b/\epsilon^2})} \\
&> w_2 \frac{[1 - \exp(-2t\sqrt{k^2 + b/\epsilon^2})]}{2}
\end{aligned}$$

$$> w_2[1-\exp(-2\epsilon_0\sqrt{k^2+b/\epsilon^2})]/2$$

$$> w_2[1-\exp(-2\sqrt{k^2\epsilon_0^2+b})]/2$$

$$> \epsilon(\sqrt{b}-k\epsilon_0)[1-\exp(-2\sqrt{k^2\epsilon_0^2+b})]/2$$

where we have used the previous bound found for w_2, the fact that the function is increasing, and we have assumed $t > \epsilon_0 > \epsilon$.

This completes the first part of the proof.

Let us show the second inequality:

$$\epsilon < \epsilon_0, \quad t \geq \epsilon_0 \Rightarrow \widetilde{P}_t \leq C_2(B,k)\epsilon.$$

Exactly as in the first part of the proof one shows

$$\widetilde{E}\{f(X)(X-\widetilde{X_t})\} \leq k\widetilde{P}_t, \quad \widetilde{E}\{\sigma^2\} \leq B.$$

From these two inequalities we deduce, by (6.10):

$$\frac{d}{dt}\widetilde{P}_t \leq 2k\widetilde{P}_t + B - \frac{1}{\epsilon^2}\widetilde{P}_t^{\,2}. \tag{6.14}$$

Now let z_t be the solution of the differential equation:

$$\dot{z} = 2kz + B - \frac{1}{\epsilon^2}z^2, \quad z_0 = \widetilde{P_0}. \tag{6.15}$$

From (6.14) and (6.15) it follows $\widetilde{P}_t \leq z_t \quad \forall t \geq 0$. Equation (6.15) is again a scalar Riccati equation. Let us solve the associated algebraic equation:

$$2kz + B - \frac{1}{\epsilon^2}z^2 = 0.$$

The solutions are given by

$$z_{1,2} = \epsilon^2(k \mp \sqrt{k^2 + \frac{B}{\epsilon^2}}).$$

Notice that z_1 is negative, whereas z_2 is positive. Moreover, this time $z_2 > -z_1$. The solution of (6.15) is

$$z = \frac{z_2 - z_1 M \exp\left(-2t\sqrt{k^2+\frac{B}{\epsilon^2}}\right)}{1 - M\exp\left(-2t\sqrt{k^2+\frac{B}{\epsilon^2}}\right)}$$

where

$$M = \frac{z_0 - z_2}{z_0 - z_1}.$$

Let us distinguish again between two cases.

(c) $z_0 > z_2$. Then it is easily seen that $0 < M < 1$ and the solution $z(t)$ is a decreasing function which approaches z_2 as $t \to \infty$. As a consequence, if we take $t > \epsilon_0$, $z(t)$ is maximized in $t = \epsilon_0$, so that

$$t > \epsilon_0 \Rightarrow \widetilde{P}_t \leq z(t) \leq z(\epsilon_0)$$

$$= \frac{z_2 - z_1 M \exp(-2\sqrt{k^2\epsilon_0^2 + B\epsilon_0^2/\epsilon^2})}{1 - M\exp(-2\sqrt{k^2\epsilon_0^2 + B\epsilon_0^2/\epsilon^2})}$$

$$< \frac{z_2 - z_1}{1 - \exp(-2\sqrt{k^2\epsilon_0^2 + B})} < \epsilon \frac{2\sqrt{k^2\epsilon_0^2 + B}}{1 - \exp(-2\sqrt{k^2\epsilon_0^2 + B})}$$

where we assumed also $\epsilon < \epsilon_0$.

(d) $z_0 \leq z_2$. In this final situation, $z(t)$ is an increasing function which approaches z_2 as $t \to \infty$, unless $z_0 = z_2$ (in this last case $z(t) = z_2$ $\forall t > 0$). Hence

$$\widetilde{P}_t \leq z(t) \leq z_2 = \epsilon(\sqrt{\epsilon^2 k^2 + B} + \epsilon k) \leq \epsilon(\sqrt{\epsilon^2 k^2} + \sqrt{B} + \epsilon k)$$

$$= \epsilon(2k\epsilon + \sqrt{B}) \leq \epsilon(2k\epsilon_0 + \sqrt{B})$$

where we assumed $\epsilon < \epsilon_0$.

The proof is finished. □

6.5 Nice behaviour with small observation noise.

We are now ready to state our main result. Consider the GPF equations (6.9) (6.10). Substitute the observation equation (6.8) in the filter equation:

$$d\widetilde{X}_t = [\widetilde{E}\{f\} - \frac{1}{\epsilon^2}\widetilde{P}_t(X_t - \widetilde{X}_t)]dt + \frac{1}{\epsilon}\widetilde{P}_t \circ dV_t. \tag{6.16}$$

Notice that the GPF itself in our case is in Itô as well as MFS form, as one can verify immediately by defining the metastate $[\widetilde{X}_t, \widetilde{P}_t]^T$ and checking with the Itô - MFS transformation (e.g. [32] page 119). So, in order to proceed, as the Itô integral has the good property that its expectation is zero, let us consider the system (6.16)-(6.10) as in Itô form. Our aim is to study the distance $E\{|X_t - \widetilde{X}_t|^2\}$ between our original system state X_t and the filter system state $\widetilde{X}_t$.

Theorem 6.5.1 *Assume (A), (B1) and (C1) are satisfied. There exist three positive real constants $L_1(b, B, k)$, $\bar{\epsilon}(b, B, k)$ and $\bar{t}(b, B, k)$ depending on the*

Lipschitz constant k of f and on the bounds b, B of σ^2 such that the following bound holds

$$E\{|X_t - \widetilde{X_t}|^2\} \leq \epsilon L_1(b,B,k), \quad \forall \epsilon \leq \bar{\epsilon}(b,B,k), \quad \forall t \geq \bar{t}(b,B,k).$$

PROOF: Let us begin by subtracting the filter equation (6.16) from the system equation (6.8):

$$\begin{aligned} d(X_t - \widetilde{X_t}) &= [f(X_t) - \widetilde{E}\{f\} - \frac{1}{\epsilon^2}\widetilde{P}_t(X_t - \widetilde{X_t})]dt \\ &+ \sigma(X_t)dW_t - \frac{1}{\epsilon}\widetilde{P}_t dV_t. \end{aligned} \tag{6.17}$$

Think of the two noise terms as a two–dimensional noise term (still standard) and apply Itô's lemma to the system (6.17) to compute

$$\begin{aligned} d[(X_t - \widetilde{X_t})^2] &= \{2(X_t - \widetilde{X_t})[f(X_t) - \widetilde{E}\{f\} \\ &- \frac{1}{\epsilon^2}\widetilde{P}_t(X_t - \widetilde{X_t})] + \sigma(X_t)^2 + \frac{1}{\epsilon^2}\widetilde{P}_t^{\,2}\}dt \\ &+ 2(X_t - \widetilde{X_t})\sigma(X_t)dW_t - \frac{2}{\epsilon}(X_t - \widetilde{X_t})\widetilde{P}_t dV_t. \end{aligned} \tag{6.18}$$

Now we can take unconditional expectation on both sides of (6.18), so that the terms representing Itô integrals vanish:

$$dE[(X_t - \widetilde{X_t})^2] = 2E\{(X_t - \widetilde{X_t})[f(X_t) - \widetilde{E}\{f\}]\}dt$$

$$+E\{\sigma(X_t)^2\}dt + \frac{1}{\epsilon^2}E\{\widetilde{P}_t^{\,2}\}dt - \frac{2}{\epsilon^2}E\{(X_t - \widetilde{X_t})^2\widetilde{P}_t\}dt. \tag{6.19}$$

From the Lipschitz assumption for f we obtain:

$$\begin{aligned} &(X_t - \widetilde{X_t})(f(X_t) - \widetilde{E}\{f\}) \\ &\leq |X_t - \widetilde{X_t}||\int (f(X_t) - f(s))p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(s)ds| \\ &\leq |X_t - \widetilde{X_t}|\int |f(X_t) - f(s)|p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(s)ds \\ &\leq |X_t - \widetilde{X_t}|\int k|X_t - s|p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(s)ds \\ &\leq |X_t - \widetilde{X_t}|\int k(|X_t - \widetilde{X_t}| + |\widetilde{X_t} - s|)p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(s)ds \end{aligned}$$

$$= k|X_t - \widetilde{X_t}|^2 + k|X_t - \widetilde{X_t}| \int |s - \widetilde{X}_t| p_{\mathcal{N}(\widetilde{X}_t, \widetilde{P}_t)}(s) ds =$$

$$= k|X_t - \widetilde{X_t}|^2 + k|X_t - \widetilde{X_t}| \sqrt{\frac{2\widetilde{P}_t}{\pi}}.$$

Let us substitute this inequality in (6.19) to obtain:

$$\begin{aligned} dE[(X_t - \widetilde{X_t})^2] &\leq 2E\{k(X_t - \widetilde{X_t})^2 + k|X_t - \widetilde{X_t}| \sqrt{\frac{2\widetilde{P}_t}{\pi}}\} dt \\ &+ E\{\sigma(X_t)^2\} dt + \frac{1}{\epsilon^2} E\{\widetilde{P_t}^2\} dt - \frac{2}{\epsilon^2} E\{(X_t - \widetilde{X_t})^2 \widetilde{P}_t\} dt. \end{aligned}$$

Now, let us use the result of Lemma (6.4.1):

$$\sqrt{\frac{2\widetilde{P}_t}{\pi}} \leq \sqrt{\frac{2C_2(B,k)\epsilon}{\pi}},$$

which combined with

$$E[|X_t - \widetilde{X_t}|] \leq (E\{(X_t - \widetilde{X_t})^2\})^{1/2}$$

yields

$$\begin{aligned} E\{|X_t - \widetilde{X_t}| \sqrt{\frac{2\widetilde{P}_t}{\pi}}\} &\leq \\ \leq \sqrt{\frac{2C_2(B,k)\epsilon}{\pi}} (E\{(X_t - \widetilde{X_t})^2\})^{1/2} &\leq \\ \leq \sqrt{\frac{2C_2(B,k)\epsilon}{\pi}} (1 + E\{(X_t - \widetilde{X_t})^2\}) & \end{aligned}$$

where we have used the inequality $\sqrt{a} < 1 + a, \forall a \geq 0$.
By including the σ^2 bounds given in assumption (B1) and using again Lemma (6.4.1) there results

$$\begin{aligned} dE[(X_t - \widetilde{X_t})^2] &\leq [2k(1 + \sqrt{\frac{2C_2(B,k)\epsilon}{\pi}}) - \frac{2}{\epsilon} C_1(b,k)] E[(X_t - \widetilde{X_t})^2] dt \\ &+ (2k\sqrt{\frac{2C_2(B,k)\epsilon}{\pi}} + B + C_2(B,k)^2) dt. \end{aligned}$$

As in the end we are interested in small ϵ, we can assume $\epsilon < 1$, from which it follows

$$\begin{aligned} dE[(X_t - \widetilde{X_t})^2] &\leq [2k(1 + \sqrt{\frac{2C_2(B,k)}{\pi}}) - \frac{2}{\epsilon} C_1(b,k)] E[(X_t - \widetilde{X_t})^2] dt + \\ &+ (2k\sqrt{\frac{2C_2(B,k)}{\pi}} + B + C_2(B,k)^2) dt. \end{aligned} \tag{6.20}$$

By setting

$$C_3(b, B, k) := 2k(1 + \sqrt{\frac{2C_2(B, k)}{\pi}})$$

$$C_4(B, k) := (2k\sqrt{\frac{2C_2(B, k)}{\pi}} + B + C_2(B, k)^2)$$

the above inequality (6.20) reads

$$dE[(X_t - \widetilde{X}_t)^2] \leq [(C_3 - 2C_1\frac{1}{\epsilon})E\{(X_t - \widetilde{X}_t)^2\} + C_4]dt.$$

Let $u(t)$ be the solution of the differential equation

$$\dot{u} = (C_3 - 2C_1\frac{1}{\epsilon})u + C_4 \tag{6.21}$$

with initial condition $u(0) = P_0$. Clearly

$$E\{(X_t - \widetilde{X}_t)^2\} \leq u(t). \tag{6.22}$$

Let us assume ϵ small enough so that the u coefficient on the right hand side of the above equation is negative:

$$\epsilon \leq \frac{2C_1}{C_3}.$$

The solution of (6.21) is

$$u(t) = [P_0 - \frac{\epsilon C_4}{(2C_1 - \epsilon C_3)}] \exp\{(C_3 - 2C_1\frac{1}{\epsilon})t\} + \\ + \frac{\epsilon C_4}{(2C_1 - \epsilon C_3)}. \tag{6.23}$$

Elementary calculations show that the exponential function appearing in (6.23) is smaller than ϵ if we take $t > t(\epsilon)$, where

$$t(\epsilon) := \frac{\log\frac{1}{\epsilon}}{2\frac{C_1}{\epsilon} - C_3} > 0.$$

By differentiating $t(\epsilon)$ w.r.t. ϵ one sees that $t(\epsilon)$ is an increasing function if $\epsilon < \exp[(-2C_1 + C_3\epsilon)/(2C_1)]$, which is implied by $\epsilon \leq \exp(-1)$. Then let us take $\epsilon \leq \exp(-1)$, so that $t(\epsilon)$ has maximum value $t(\exp(-1))$. Hence

$$\begin{aligned} t &> t(\exp(-1)) \Rightarrow t > t(\epsilon) \Rightarrow \exp[(C_3 - 2C_1/\epsilon)] < \epsilon \\ &\Rightarrow u(t) < [P_0 - \frac{\epsilon C_4}{(2C_1 - \epsilon C_3)}]\epsilon + \frac{\epsilon C_4}{(2C_1 - \epsilon C_3)} \\ &\Rightarrow u(t) < \epsilon[P_0 + \frac{C_4}{(2C_1 - \epsilon C_3)}]. \end{aligned}$$

Now choose any positive real number $\bar{\epsilon}$ and the positive real number $\bar{t}$ according to

$$
\begin{aligned}
\bar{\epsilon} &< \min\{\exp(-1), 2C_1/C_3, \epsilon_0\} , \\
\bar{t} &= \max\{t(\exp(-1)), \epsilon_0\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
t &> \bar{t}, \;\; \epsilon < \bar{\epsilon} \;\Rightarrow\; u(t) < \epsilon L_1 , \\
L_1 &= [P_0 + \frac{C_4}{(2C_1 - \bar{\epsilon}C_3)}]
\end{aligned}
$$

and (6.22) completes the proof. □

From the theorem above the following corollary follows easily.

Corollary 6.5.2 *Assume (A), (B1) and (C1) are satisfied. For every sequence ϵ_n, $n \in I\!N$ such that $0 < \epsilon_n < \tilde{\epsilon}(b, B, k)$ for all $n \in I\!N$ and $\sum_{n=1}^{\infty} \epsilon_n < +\infty$ we have*

$$
\frac{1}{T-\bar{t}} \int_{\bar{t}}^{T} |X_t - \widetilde{X}_t^{\epsilon_n}|^2 dt \to 0 \;\; a.s.
$$

as $n \to \infty$ and for every $T > \bar{t}$.

PROOF: From the theorem above, Chebychev inequality and the mean value theorem for integrals we obtain

$$
P\{\frac{1}{T-\bar{t}} \int_{\bar{t}}^{T} |X_t - \widetilde{X}_t^{\epsilon_n}|^2 dt > \delta\} \leq \frac{L_1 \epsilon_n}{\delta}
$$

for every $\delta > 0$ and we can conclude by Borel–Cantelli's lemma. In fact

$$
P[\, \limsup_n \{\frac{1}{T-\bar{t}} \int_{\bar{t}}^{T} |X_t - \widetilde{X}_t^{\epsilon_n}|^2 dt > \delta\} \,] = 0
$$

which concludes the proof.

□

The above results ensure that the state X_t of the system is well estimated by $\widetilde{X}_t^{\epsilon}$ for small observation noise ϵ.

6.6 One–dimensional efficient GPF

Picard showed in [51] that, under suitable assumptions, there exists a *one–dimensional* filter whose mean–square difference from the true state is bounded

by a constant which is proportional to the magnitude of the observation noise. In this section we show that there exists a *one–dimensional* GPF which is as efficient as the two–dimensional one (6.9-6.10), in the sense that they yield the same mean–square error bound when compared to the true state.
In this section we shall use the geometric derivation of the projection filter. This is the only part of the chapter where we invoke geometry. We repeat the derivation because we prefer to keep the chapter as self contained as possible, and also to notice what happens when projecting the Duncan–Mortensen–Zakai equation for nonlinear filtering instead of the Kushner–Stratonovich equation given in Chapter 2.

Consider the Duncan–Mortensen–Zakai stochastic partial differential equation (DMZ) for an unnormalized version $q_t(\cdot)$ of the optimal–filter density $p_t(\cdot)$ (where $p_t(x)dx := P\{X_t \in dx|\mathcal{Y}_t\}$). For the system (6.8) such partial SDE in MFS form reads

$$d_t q_t(x) = (\mathcal{L}^* q_t)(x)dt - \tfrac{1}{2}x^2 q_t(x)dt + x q_t(x) \circ dY_t \tag{6.24}$$

(this can be obtained for example by formula (40) page 72 of [19] with an integration by parts). Projection of this equation results in the same filter as in the case of projection of the Kushner–Stratonovich equation. This is due to the fact that since we are projecting on a manifold of *normalized* densities, the projection automatically takes care of the normalization step.

In order to use the L_2 inner product, consider the DMZ equation for the square root of $q_t(\cdot)$:

$$d_t\sqrt{q_t(x)} = \frac{(\mathcal{L}^* q_t)(x)}{2\sqrt{q_t(x)}}dt - \tfrac{1}{2}x^2\frac{\sqrt{q_t(x)}}{2}dt + x\frac{\sqrt{q_t(x)}}{2} \circ dY_t. \tag{6.25}$$

Next, select a finite–dimensional manifold of square roots of densities to approximate the optimal filter $\sqrt{p_t(\cdot)}$. Let the family be parametrized by $\theta \in \Theta \subseteq \mathbf{R}^m$, where Θ is open. Call such manifold $S^{1/2}$,

$$S^{1/2} = \{\sqrt{p(\cdot,\theta)}, \theta \in \Theta\}.$$

Consider a generic curve $t \mapsto \sqrt{p(\cdot,\theta_t)}$ on this manifold. Its tangent vector in θ_t is given according to the chain rule:

$$d_t\sqrt{p(\cdot,\theta_t)} = \sum_{i=1}^{m} \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i} \circ d\theta_t^i\ , \tag{6.26}$$

from which we see that tangent vectors in θ_t to all curves lie in the linear (tangent) space

$$\mathrm{span}\{\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_1},\ldots,\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_m}\}.$$

Consider the Fisher information

$$g(\theta)_{ij} := 4\langle \frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_i}, \frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_j}\rangle, \quad i,j=1,\dots,m,$$

where $\langle\cdot,\cdot\rangle$ is the inner product of L_2 (see Chapter 3 for the details). Now consider for all $\theta\in\Theta$ the orthogonal projection

$$\Pi_\theta : L_2 \longrightarrow \operatorname{span}\{\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_1},\dots,\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_m}\}$$

$$\Pi_\theta[v] := \sum_{i=1}^m [\sum_{j=1}^m 4g^{ij}(\theta)\,\langle v, \frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_j}\rangle]\ \frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_i}\ .$$

At this point we project the DMZ equation (6.25) via this projection, obtaining the following (m–dimensional) SDE on the manifold $S^{1/2}$:

$$d\sqrt{p(\cdot,\theta_t)} \;=\; \Pi_{\theta_t}[\frac{\mathcal{L}^* p(\cdot,\theta_t)}{2\sqrt{p(\cdot,\theta_t)}}]dt - \tfrac{1}{2}\Pi_{\theta_t}[x^2\frac{\sqrt{p(\cdot,\theta_t)}}{2}]dt + \Pi_{\theta_t}[x\frac{\sqrt{p(\cdot,\theta_t)}}{2}]\circ dY_t.$$

Writing the projection map explicitly and comparing with (6.26) yields the following SDE for the parameters:

$$\begin{aligned} d\theta_t \;=\;& g^{-1}(\theta_t)\,[\int \frac{\mathcal{L}_t^* p(x,\theta_t)}{p(x,\theta_t)}\,\frac{\partial p(x,\theta_t)}{\partial\theta}\,dx]\,dt \\ & -g^{-1}(\theta_t)[\int \tfrac{1}{2}x^2\frac{\partial p(x,\theta_t)}{\partial\theta}\,dx]\,dt + \\ & +g^{-1}(\theta_t)\,[\int x\,\frac{\partial p(x,\theta_t)}{\partial\theta}\,dx]\circ dY_t, \end{aligned} \tag{6.27}$$

where integrals of vector functions are meant to be applied to their components. As we saw in Chapter 4, the equation above can be simplified by selecting suitable exponential manifolds involving the observation function h (identity in our case) and its square. Such manifolds are called $S_\bullet^{1/2}$ (involving both h and h^2) and $S_*^{1/2}$ (involving only h). In our case the manifold $S_\bullet^{1/2}$ turns out to be the Gaussian manifold (exponential with second degree polynomials in the exponent, i.e. combinations of $h=x$ and $h^2=x^2$) and hence the projection filter is a GPF. This is actually the two–dimensional filter which coincides with the MFS–G–ADF (as seen in Chapter 5) and on which we worked so far in this chapter. Lemma (6.4.1), Theorem (6.5.1) and Corollary (6.5.2) all concern this filter.

Now we decide to use instead the manifold $S_*^{1/2}$, i.e. we involve only $h=x$ in the exponent of the exponential manifold. We can then fix the coefficient of the second degree term in the manifold (which amounts to fix the variance) by defining

$$S_*^{1/2} \;:=\; \{\sqrt{p(\cdot,\theta)} : \theta \in \mathbf{R}\},$$

$$p(x,\theta) \;:=\; \exp(\theta x - \tfrac{1}{2\epsilon}x^2 - \psi(\theta)) = p_{\mathcal{N}(\epsilon\theta,\epsilon)}(x).$$

Notice that the smaller the noise ϵ, the smaller the variance. The normalizing constant $\psi(\theta)$ is given by

$$\psi(\theta) = \tfrac{1}{2}\log(2\pi\epsilon) + \tfrac{1}{2}\epsilon\theta^2.$$

This is clearly a one–dimensional Gaussian manifold. This case is slightly different from the case treated in Chapter 4, since here the coefficient of x^2 in the exponent is not a parameter. But all the relevant facts for exponential manifolds seen in Chapter 3 still hold. Consider now the expectation parameter as defined in Chapter 3 :

$\bar{X}(\theta) := \frac{d\psi(\theta)}{d\theta} = \epsilon\theta$, and notice that of course $p(\cdot,\theta) = p_{\mathcal{N}(\bar{X},\epsilon)}(\cdot)$. The Fisher information can be computed as in Lemma 3.3.3: $g(\theta) := \frac{d\bar{X}(\theta)}{d\theta} = \epsilon$. Denote by both E_θ and $E_{\bar{X}}$ expectation w.r.t. the Gaussian density $p(\cdot,\theta)$. Observe that by the well–known formulas for moments of Gaussian random variables we have $E_\theta\{\frac{1}{2}X^2(X-\bar{X})\} = \bar{X}$. Notice that $\mathcal{L}X = f$. Now we can write the projection filter equation relative to the manifold S_*. Then equation (6.27) particularizes to

$$d\theta_t = \tfrac{1}{\epsilon}E_{\theta_t}\{f\}\;dt - \tfrac{1}{\epsilon}\theta_t\;dt + \tfrac{1}{\epsilon^2}\circ dY_t$$

or, w.r.t. the expectation parameter

$$d\bar{X}_t = E_{\bar{X}_t}\{f\}dt - \tfrac{1}{\epsilon}\bar{X}_t\;dt + \tfrac{1}{\epsilon}\circ dY_t.$$

The above SDE is valid both in the Itô and in the MFS sense. Compute by Itô's lemma $d[(X_t-\bar{X}_t)^2]$ and take expectation on both sides (expectations of Itô integrals vanish):

$$\begin{aligned} dE\{(X_t-\bar{X}_t)^2\} \;&=\; 2E\{(X_t-\bar{X}_t)(f(X_t)-E_{\bar{X}_t}\{f\})\}dt \\ &-\; \frac{2}{\epsilon}E\{(X_t-\bar{X}_t)^2\}dt + E\{1+\sigma(X_t)^2\}dt. \end{aligned}$$

By the same arguments used in the proof of Theorem (6.5.1) and straightforward calculations we obtain the following differential inequality:

$$\frac{d}{dt}E\{(X_t-\bar{X}_t)^2\} \;\leq\; (2k + k\sqrt{\frac{2\epsilon}{\pi}} - \frac{2}{\epsilon})E\{(X_t-\bar{X}_t)^2\}$$

$$+ \quad k\sqrt{\frac{2\epsilon}{\pi}} + 1 + B.$$

Starting from this last inequality, computations similar to those given in the previous section yield the following theorem.

Theorem 6.6.1 *Assume (A), (B1) and (C1) are satisfied. Choose the three constants $\epsilon(k)$, $T(k)$ and $C(k)$ according to*

$$\epsilon(k) < \min(1, \pi[1 + \tfrac{1}{k}\sqrt{\frac{2}{\pi}} - \sqrt{1 + \frac{2}{k}\sqrt{\frac{2}{\pi}}}\ ]),$$

$$T(k) = 1/[2 - k\ \epsilon(k)(2 + \sqrt{\frac{2\ \epsilon(k)}{\pi}}\ )]$$

$$C(k) = (1 + B + k\sqrt{\frac{2\ \epsilon(k)}{\pi}}\ )\ T(k).$$

The following bound holds:

$$E\{(X_t - \bar{X}_t)^2\} \leq [E\{(X_0 - \bar{X}_0)^2\} + C(k)]\epsilon,$$

$$\forall \epsilon < \epsilon(k),\ \forall t > T(k).$$

Finally, Corollary (6.5.2) can be easily translated for this one–dimensional filter:

Corollary 6.6.2 *For every sequence ϵ_n, $n \in I\!N$ such that $0 < \epsilon_n < \epsilon(k)$ for all n and $\sum_{n=1}^{\infty} \epsilon_n < +\infty$ we have*

$$\frac{1}{T - T(k)} \int_{T(k)}^{T} |X_t - \bar{X}_t^{\ \epsilon_n}|^2 dt \rightarrow 0\ a.s. \tag{6.28}$$

as $n \rightarrow \infty$ and for every $T > T(k)$.

6.7 Extension of the results to different models.

So far we have worked under assumptions (A), (B1) and (C1) given in the beginning of the chapter. As far as the comparison true state–GPF is concerned, the results obtained so far can be extended to systems like (6.1) satisfying different assumptions. In this section we still assume (B1) to hold (σ^2 uniformly bounded). Moreover, we replace assumptions (A) and (C1) by (D) and (E):

(D) We assume f bounded.

(E) We assume h bijective, $h \in C^2$, and $|h_x|$ uniformly bounded. Moreover, we assume h_{xx}^2 bounded.

In the present section we shall extend Corollary (6.6.2) to this situation. The extension of Corollary (6.5.2) is analogous. The main change regards the drift f. As long as the drift is bounded, we do not need it to be Lipschitz continuous.

Theorem 6.7.1 *Assume (B1), (D) and (E) are satisfied. For every sequence ϵ_n, $n \in I\!N$ such that $0 < \epsilon_n < 1$ for all n and $\sum_{n=1}^{\infty} \epsilon_n < +\infty$, let $\{\check{\xi}_t^{\epsilon_n},\ t \geq 0\}$, $n \in I\!N$ be the sequence of stochastic processes defined by the SDEs*

$$d\check{\xi}_t^{\epsilon_n} = -\tfrac{1}{\epsilon_n}\check{\xi}_t^{\epsilon_n}\, dt + \tfrac{1}{\epsilon_n} \circ dY_t^{\epsilon_n}, \quad \check{\xi}_0^{\epsilon_n} = E\{h(X_0)\}$$

Then we have

$$\frac{1}{T-\frac{1}{2}} \int_{\frac{1}{2}}^{T} |X_t - h^{-1}(\check{\xi}_t^{\epsilon_n})|^2 dt \to 0 \ a.s.$$

as $n \to \infty$ and for every $T > \frac{1}{2}$.

Proof: We use Girsanov's theorem to modify the state equation by changing probability measure (see for example [25]). Fix an arbitrary $T > \frac{1}{2}$ and set

$$\begin{aligned} \phi &:= -\frac{h_{xx}\, \sigma^2}{2\ h_x}, \\ z_t &:= \frac{\phi(X_t) - f(X_t)}{\sigma(X_t)}, \ \tfrac{1}{2} \leq t \leq T. \end{aligned}$$

Define the new probability measure P_1 according to

$$\alpha_t := \exp\{\int_{\frac{1}{2}}^{t} z_s dW_s - \tfrac{1}{2}\int_{\frac{1}{2}}^{t} z_s^2 ds\},$$

$$P_1(A) := \int_A \alpha_T(\omega) P(d\omega), \quad A \in \mathcal{F}.$$

It is easy to check that under assumptions (B), (D) and (E) there results

$$E\{\exp[\tfrac{1}{2}\int_{\frac{1}{2}}^{T} |z_s|^2 ds]\} < \infty,$$

so that we can apply the measure transformation above and use Girsanov's theorem. Under P_1, the system equation (6.1) becomes

$$dX_t = \phi(X_t)\, dt + \sigma(X_t) dW_t^1,$$

where $dW_t^1 := dW_t - [\phi(X_t) - f(X_t)]/\sigma(X_t)dt$ is a standard Brownian motion. Now, the function ϕ has been purposefully defined in such a way that, by applying Itô's lemma, one obtains

$$dh(X_t) = h_x(X_t)\sigma(X_t)dW_t^1.$$

Set $\xi_t := h(X_t),\ t \geq 0$. Rewrite (6.1) for $h(X_t)$ under P_1:

$$\begin{aligned} d\xi_t &= h_x(h^{-1}(\xi_t))\sigma(h^{-1}(\xi_t))dW_t^1, \\ dY_t &= \xi_t dt + \epsilon dV_t. \end{aligned} \tag{6.29}$$

Consider now the system (6.29). Notice that the diffusion coefficient of the state equation is uniformly bounded according to assumptions (C1) and (E) : $0 < H_1\sqrt{b} \leq |\sigma h_x| \leq H_2\sqrt{B}$. Moreover, the observation function is the identity function. The state equation is particularly simple, since its drift is zero. As a consequence, the drift is trivially uniformly Lipschitz with arbitrary constant k. Write the one–dimensional GPF for the system (6.29):

$$d\check{\xi}_t^\epsilon = -\tfrac{1}{\epsilon}\check{\xi}_t^\epsilon\, dt + \tfrac{1}{\epsilon} \circ dY_t^\epsilon, \quad \check{\xi}_0 = E\{h(X_0)\}.$$

Apply then Corollary 6.6.2 with $k \to 0$. We obtain

$$\tfrac{1}{T-\frac{1}{2}} \int_{\frac{1}{2}}^{T} |\xi_t - \check{\xi}_t^{\epsilon_n}|^2 dt \to 0, \quad P_1 - a.s.$$

as $n \to \infty$. Since $P_1 \sim P$, the last result holds also $P-$ a.s. Uniform boundedness of $|h_x|$ implies that h^{-1} is Lipschitz with constant $1/H_1$. Hence

$$|X_t - h^{-1}(\check{\xi}_t^{\epsilon_n})|^2 \leq |\xi_t - \check{\xi}_t^{\epsilon_n}|^2/H_1^2,$$

for all t in $[\frac{1}{2}\ ,T]$. The theorem is proved. $\square$

Notice that the filter given in this last theorem does not depend on the drift term f of the state equation.

6.8 GPF versus optimal filter

In this section we go back to the two–dimensional GPF given in equations (6.9, 6.10) and we assume (A), (B1), and (C1) to be in force. We present a theorem dealing with the quality of the approximation obtained with the GPF with respect to the optimal filter instead of the signal (true state). We consider the mean–square difference $E\{|\widehat{X}_t - \widetilde{X}_t|^2\}$ between the optimal filter estimate $\widehat{X}_t$ and the GPF estimate $\widetilde{X}_t$, and we average it over a time interval $[\bar{t}\quad T]$, T arbitrarily large. The corresponding time average

$$\frac{1}{T-\bar{t}} \int_{\bar{t}}^{T} E\{|\widehat{X}_t - \widetilde{X}_t|^2\}dt$$

will be proved to be bounded by a quantity proportional to ϵ^2. More precisely, the following theorem holds:

Theorem 6.8.1 *Assume (A), (B1) and (C1) are satisfied. Let $\bar{\epsilon}$ and $\bar{t}$ be given according to Theorem 6.5.1. There exists a positive real number $\tilde{\epsilon} < \bar{\epsilon}$ such that*

$$\epsilon \leq \tilde{\epsilon}, \quad T > \bar{t} \Rightarrow \quad \exists t_1 \in [\bar{t}, T], \quad \exists M(\bar{t}, t_1) \in \mathbf{R}^+ :$$

$$\frac{1}{T-\bar{t}} \int_{\bar{t}}^{T} E\{|\widehat{X}_t - \widetilde{X}_t|^2\} dt < M(\bar{t}, t_1)\epsilon^2. \tag{6.30}$$

PROOF: Let us fix an arbitrary $T > \bar{t}$. From now on, unless differently specified, all time instants are meant to belong to the time interval $\mathcal{T} := [\bar{t}, \ T]$. We begin by defining the stochastic process $c_t := X_t - \widetilde{X}_t$. From (6.8, 6.9) one sees that such a process satisfies the following SDE:

$$\begin{aligned} dc_t &= [f(c_t + \widetilde{X}_t) - \widetilde{E}\{f\}]dt + \\ &- \tfrac{1}{\epsilon^2}\widetilde{P}_t c_t dt - \tfrac{1}{\epsilon}\widetilde{P}_t dV_t + \sigma(X_t) dW_t. \end{aligned} \tag{6.31}$$

From such SDE it is clear that c_t depends on the state of the system only via the last term. Then let us consider an approximation of the process c_t which does not depend on the state X_t. Define the process γ_t for $t \geq \bar{t}$ as the solution of the following SDE:

$$\begin{aligned} d\gamma_t &= [f(\gamma_t + \widetilde{X}_t) - \widetilde{E}\{f\}]dt - \tfrac{1}{\epsilon^2}\widetilde{P}_t \gamma_t dt - \tfrac{1}{\epsilon}\widetilde{P}_t dV_t + Q_t dW_t, \\ \gamma_{\bar{t}} &\sim \mathcal{N}(0, \widetilde{P}_{\bar{t}}) \end{aligned} \tag{6.32}$$

where we replaced $\sigma(X_t)$ with the quantity $Q_t := \sqrt{\widetilde{E}\{\sigma^2\} - \epsilon}$, which we can assume well defined (see assumption (B1)), provided that ϵ is small enough. Now we consider the difference between c_t and its approximation γ_t. Define

$$s_t := (c_t - \gamma_t) \ \mathbf{1}_{\mathcal{T}}(t), \quad t \geq 0,$$

where $\mathbf{1}_{\mathcal{T}}$ is the indicator function of the set $\mathcal{T}$. Now we define an approximation of the process $\{X_t, \quad t \in \mathcal{T}\}$. Since $X_t = \widetilde{X}_t + c_t$, we define its approximation χ_t based on the approximation of c_t via γ_t:

$$\chi_t := \widetilde{X}_t + \gamma_t.$$

Notice that $\chi_{\bar{t}}|\mathcal{Y}_{\bar{t}} \sim \mathcal{N}(\widetilde{X}_{\bar{t}}, \widetilde{P}_{\bar{t}})$. From the SDE of $\widetilde{X}_t$, γ_t and from the definition of s_t it follows easily that

$$d\chi_t = [\ \tfrac{1}{\epsilon^2}\widetilde{P}_t s_t + \ f(\chi_t)]\ dt + Q_t\ dW_t. \tag{6.33}$$

The observation process y can be written as

$$dY_t = \ (s_t + \chi_t)\ dt + \ \epsilon\ dV_t. \tag{6.34}$$

At this point, we consider the system with state equation (6.33) and observations (6.34). We plan to use Girsanov's theorem to define a new probability measure under which the SDE for χ becomes linear and the s_t term in equation (6.34) vanishes. Consider the following SDE, expressing (6.33,6.34) in vector form:

$$d\begin{bmatrix} \chi_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \frac{1}{\epsilon^2}\widetilde{P}_t s_t + f(\chi_t) \\ \chi_t + s_t \end{bmatrix} dt + \begin{bmatrix} Q_t & 0 \\ 0 & \epsilon \end{bmatrix} d\begin{bmatrix} W_t \\ V_t \end{bmatrix}.$$

Define

$$\psi_t := \begin{bmatrix} Q_t & 0 \\ 0 & \epsilon \end{bmatrix}^{-1} \left(\begin{bmatrix} a_t\chi_t + b_t \\ \chi_t \end{bmatrix} - \begin{bmatrix} \frac{1}{\epsilon^2}\widetilde{P}_t s_t + f(\chi_t) \\ \chi_t + s_t \end{bmatrix} \right)$$

where a_t and b_t are two predictable processes (in particular they need to be adapted: a_t and b_t are $\mathcal{Y}_t$–measurable for all $t \geq 0$) which will be introduced later. Let us denote by ζ_t the first component of ψ_t, i.e.

$$\zeta_t := \frac{1}{Q_t}[a_t\chi_t + b_t - \ f(\chi_t) - \ \frac{\widetilde{P}_t}{\epsilon^2}s_t]$$

so that

$$\psi_t = \begin{bmatrix} \zeta_t \\ -s_t/\epsilon \end{bmatrix}.$$

Define the new probability measure P_0 on $(\Omega, \mathcal{F})$ according to the following formula:

$$\rho_t := \exp\{\int_{\bar{t}}^t \psi_s^T d\begin{bmatrix} W_s \\ V_s \end{bmatrix} - \frac{1}{2}\int_{\bar{t}}^t |\psi_s|^2 ds\},$$

$$P_0(A) := \int_A \rho_T(\omega)\ P(d\omega), \quad A \in \mathcal{F}. \tag{6.35}$$

The new probability measure P_0 is well defined if

$$E\{\exp[\frac{1}{2}\int_{\bar{t}}^T |\psi_s|^2 ds]\} < +\infty. \tag{6.36}$$

We shall prove later that this condition is satisfied. Girsanov's theorem states that under P_0 the stochastic process

$$\begin{bmatrix} W_t^0 \\ V_t^0 \end{bmatrix} := \begin{bmatrix} W_t \\ V_t \end{bmatrix} - \int_{\bar{t}}^t \psi_s ds\ , \quad t \in \mathcal{T}$$

is a standard Brownian motion. Moreover, in $(\Omega, \mathcal{F}, P_0)$ the process $[\chi_t \quad Y_t]^T$ satisfies the following SDE:

$$d\begin{bmatrix} \chi_t \\ Y_t \end{bmatrix} = \begin{bmatrix} a_t\chi_t + b_t \\ \chi_t \end{bmatrix} dt + \begin{bmatrix} Q_t & 0 \\ 0 & \epsilon \end{bmatrix} d\begin{bmatrix} W_t^0 \\ V_t^0 \end{bmatrix}. \tag{6.37}$$

Now, consider this last system as a state process χ_t with observations Y_t. The system is linear and at time $\bar{t}$ it has a Gaussian initial condition. As a consequence, the optimal filter is described by a Gaussian density. Now, if we project such filter onto the tangent space of a manifold of Gaussian densities (cf. Chapter 4), the filter does not change. This implies that for this system the GPF is equal to the optimal filter. Then the GPF yields exactly the two quantities $\widehat{\chi}_t^0 := E_0\{\chi_t|\mathcal{Y}_t\}$, $P_t^0 := E_0\{(\chi_t - \widehat{\chi}_t^0)^2|\mathcal{Y}_t\}$, where E_0 denotes expectation w.r.t. the probability measure P_0. On the other hand, let us derive the equations of the GPF for the system (6.37) above. This can be done via the assumed–density principle (see Chapter 5 and Section 6.2). We obtain

$$\begin{aligned} d\widehat{\chi}_t^0 &= [a_t\widehat{\chi}_t^0 + b_t - \tfrac{1}{\epsilon^2}P_t^0\widehat{\chi}_t^0]\, dt + \tfrac{1}{\epsilon^2}P_t^0 \circ dY_t \\ \\ dP_t^0 &= [2E_0\{(a_t\chi_t + b_t)(\chi_t - \widehat{\chi}_t^0)|\mathcal{Y}_t\} + Q_t^2 - \tfrac{1}{\epsilon^2}(P_t^0)^2]\, dt. \end{aligned}$$

Now we choose a_t and b_t in such a way that by substituting $\widehat{\chi}_t^0 = \widetilde{X}_t$ and $P^0 = \widetilde{P}_t$ in these last equations they become equations (6.9, 6.10). This can be done by choosing

$$\begin{aligned} a_t &:= \frac{2\widetilde{E}\{f(x)\ (x - \widetilde{X}_t)\} + \epsilon}{2\widetilde{P}_t}, \\ \\ b_t &:= \widetilde{E}\{f\} - a_t\widetilde{X}_t. \end{aligned}$$

Then, from uniqueness of the solution,

$$\widehat{\chi}_t^0 = \widetilde{X}_t, \quad P_t^0 = \widetilde{P}_t. \tag{6.38}$$

Now compute by Itô's formula

$$d\rho_t = \zeta_t\ \rho_t\ dW_t - \rho_t\ \frac{s_t}{\epsilon}\ dV_t. \tag{6.39}$$

Let us consider the filtering problem with state equation (6.39) and observation process Y_t satisfying equation (6.8) (remember that both ζ and s depend on X). We compute the optimal filter for this system via formula (8.10) page 299 of [45]. There results the following filter:

$$d\widehat{\rho}_t = \tfrac{1}{\epsilon}\ [-\pi_t(\rho_t\ s_t) + \pi_t(X_t\rho_t) - \widehat{X}_t\widehat{\rho}_t]\ d\nu_t,$$

$$d\nu_t = \tfrac{1}{\epsilon}[dY_t - \widehat{X}_t dt],$$

$$\nu_{\bar{t}} = Y_{\bar{t}} - \int_0^{\bar{t}} \widehat{X}_s ds. \tag{6.40}$$

Kallianpur-Striebel formula and (6.38) yield

$$\begin{aligned} \pi_t(X_t\rho_t) &= E_0\{s_t + \chi_t|\mathcal{Y}_t\}\ \widehat{\rho}_t = (\widehat{s}_t^0 +\ \widehat{\chi}_t^0)\ \widehat{\rho}_t \\ &= (\frac{\pi_t(s_t\rho_t)}{\widehat{\rho}_t} + \widetilde{X}_t)\widehat{\rho}_t = \pi_t(s_t\ \rho_t) +\ \widehat{\rho}_t\ \widetilde{X}_t. \end{aligned}$$

Substitution of this last result in (6.40) yields the following SDE

$$d\widehat{\rho}_t = \tfrac{1}{\epsilon}\widehat{\rho}_t(\widetilde{X}_t - \widehat{X}_t)d\nu_t,$$

whose solution at time T is

$$\widehat{\rho}_T = \exp\{\tfrac{1}{\epsilon}\int_{\bar{t}}^T (\widetilde{X}_s - \widehat{X}_s)d\nu_s - \tfrac{1}{2\epsilon^2}\int_{\bar{t}}^T |\widetilde{X}_s - \widehat{X}_s|^2 ds\}. \tag{6.41}$$

Jensen's inequality gives $-\log \widehat{\rho}_T \leq -E\{\log \rho_T\ |\mathcal{Y}_T\}$, from which, taking expectations,
$-E\{\log \widehat{\rho}_T\} \leq -E\{E\{\log \rho_T\ |\mathcal{Y}_T\}\}$ which reads (remembering (6.35), (6.41) and the fact that expectations of Itô integrals vanish)

$$\tfrac{1}{2\epsilon^2}\int_{\bar{t}}^T E\{|\widetilde{X}_u - \widehat{X}_u|^2\}du \leq \tfrac{1}{2}\int_{\bar{t}}^T E\{\zeta_u^2\}du + \tfrac{1}{2\epsilon^2}\int_{\bar{t}}^T E\{s_u^2\}du$$

which implies

$$\int_{\bar{t}}^T E\{|\widetilde{X}_u - \widehat{X}_u|^2\}du \leq \epsilon^2(\int_{\bar{t}}^T E\{\zeta_u^2\}du + \tfrac{1}{\epsilon^2}\int_{\bar{t}}^T E\{s_u^2\}du). \tag{6.42}$$

Now we continue the proof with the two following facts **a)** and **b)**:

a) The second integral in the above inequality satisfies the following estimate:

$$\int_{\bar{t}}^T E\{s_u^2\}du \leq L_2(\bar{t},T)\epsilon^2, \tag{6.43}$$

where $L_2(\bar{t},T)$ is a positive real constant.
We begin the proof of this estimate by showing that

$$E\{s_{\bar{t}}^2\} \leq C_3\epsilon,$$

where $C_3 := C_2(B,k) + L_1(b,B,k)$. This is immediate from

$$E\{s_{\bar{t}}^2\} = E\{(c_{\bar{t}} - \gamma_{\bar{t}})^2\} \leq 2E\{c_{\bar{t}}^2\} + 2E\{\gamma_{\bar{t}}^2\} \leq 2(C_2(B,k) + L_1(b,B,k))\epsilon,$$

where the last two inequalities follow from the fact that $\gamma_{\bar{t}} \sim \mathcal{N}(0, \widetilde{P}_{\bar{t}})$, from Lemma 6.4.1 and from Theorem 6.5.1. Consider then the following SDE for $\{s_t, t \in \mathcal{T}\}$,

$$\begin{aligned} d\, s_t &= d\, c_t - d\, \gamma_t = \\ &= [f(s_t + \chi_t) - f(\chi_t)]\, dt - \tfrac{1}{\epsilon^2}\widetilde{P}_t\, s_t\, dt + [\sigma(X_t) - Q_t]\, dW_t. \end{aligned}$$

By Itô's formula

$$\begin{aligned} d\, s_t^2 &= 2s_t\, [f(s_t + \chi_t) - f(\chi_t) - \tfrac{1}{\epsilon^2}\widetilde{P}_t\, s_t]\, dt \\ &+ [\sigma(X_t) - Q_t]^2\, dt + 2s_t\, [\sigma(X_t) - Q_t]\, dW_t. \end{aligned} \tag{6.44}$$

Lipschitz assumptions on f yield $|f(s_t+\chi_t) - f(\chi_t)| \leq k|s_t|$, and by definition of Q_t we see that $[\sigma(X_t) - Q_t]^2 \leq (|\sigma(X_t)| + |Q_t|)^2 \leq (\sqrt{B} + B)^2$, where we used the uniform boundedness of σ^2. By substituting these last inequalities in (6.44), using Lemma (6.4.1) for $\widetilde{P}_t$ and taking expectations we obtain the following differential inequality, which holds also for $t < \bar{t}$:

$$\frac{d}{dt}\, E\{s_t^2\} \leq 2\, (k - \tfrac{1}{\epsilon^2}C_2)\, E\{s_t^2\} + (\sqrt{B} + B)^2. \tag{6.45}$$

By using usual estimates for differential inequalities (see for example the final part of the proof of Theorem (6.5.1)), we can easily conclude that there exists a positive real number $\tilde{\epsilon} < \bar{\epsilon}$ small enough and a positive constant C_4 such that

$$E\{s_t^2\} < C_4\epsilon^2 \tag{6.46}$$

for all $\epsilon < \tilde{\epsilon}$ and $t \in \mathcal{T}$. If necessary, one can increase $\bar{t}$ to ensure (6.46), and the proof still holds. Integration of this last inequality yields (6.43).

b) The first integral in the r.h.s. of inequality (6.42) behaves nicely with respect to ϵ, in the sense that it can be bounded by a constant independent of ϵ.

In order to prove this, notice that

$$\begin{aligned} \zeta_t \leq \tfrac{1}{Q_t}(|\widetilde{E}\{f\} - f(\chi_t)| &+ \tfrac{1}{\epsilon^2}\widetilde{P}_t|s_t| \\ &+ \frac{|2\widetilde{E}\{f(X_t)(X_t - \widetilde{X}_t)\} + \epsilon|}{2\widetilde{P}_t}\, |\chi_t - \widetilde{X}_t|\,). \end{aligned} \tag{6.47}$$

Now observe that the following relations hold:

$$\widetilde{E}\{f(X)(X-\widetilde{X}_t)\} \leq k\widetilde{P}_t$$

(see proof of Lemma (6.4.1));

$$|\widetilde{E}\{f\} - f(\chi_t)| \leq k\ (|\widetilde{X}_t - \ \chi_t| + \sqrt{2\ \widetilde{P}_t/\pi}) = k\ (|\gamma_t| + \sqrt{2\ \widetilde{P}_t/\pi})$$

(analogous to proof of Theorem (6.5.1));

$$|\chi_t - \widetilde{X}_t| \leq\ |\chi_t -\ X_t| +\ |X_t - \widetilde{X}_t| = |s_t| +\ |X_t - \widetilde{X}_t|;$$

and, finally,

$$1/Q_t \leq 1/\sqrt{b-\epsilon}$$

(by definition of Q_t and bounds for σ^2).

By substituting such relations in (6.47) and using Lemma (6.4.1) again, we obtain

$$\begin{aligned}\zeta_t \quad &\leq \quad \tfrac{1}{\sqrt{b-\epsilon}}\{k\ (|\gamma_t| + \sqrt{2\ C_2\epsilon/\pi})\\ &+ \quad \tfrac{1}{\epsilon}C_2|s_t| + \ \frac{1+2kC_2}{2C_1}(|s_t| + |X_t - \widetilde{X}_t|)\}.\end{aligned}$$

By regrouping terms, assuming $\epsilon \leq \tilde{\epsilon}$ and observing that in general $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$ for all $a,b,c,d\ \in \mathbf{R}$, we obtain

$$\begin{aligned}\zeta_t^2 \leq \frac{8}{(b-\tilde{\epsilon})}\{k^2\gamma_t^2 + 2\frac{C_2}{\pi}k^2\tilde{\epsilon} + (\frac{C_2}{\epsilon} + \frac{1+2kC_2}{2C_1})^2 s_t^2\\ +(\frac{1+2kC_2}{2C_1})^2(X_t - \widetilde{X}_t)^2\}.\end{aligned} \tag{6.48}$$

Now we can take expectations on both sides of the above inequality. Consider then the four terms in the right hand side. The second one does not depend on ϵ, so that we do not need to consider it. The third one has a coefficient which is partly divided by ϵ^2; however, since this coefficient is multiplied by $E\{s_t^2\}$, the relation (6.46) ensures that such central term behaves nicely w.r.t. small ϵ. For the first term, the ϵ depending part is $E\{\gamma_t^2\}$. Observe that for $t \in \mathcal{T}$ and $\epsilon < \tilde{\epsilon}$ we have

$$E\{\gamma_t^2\} = E\{(c_t - s_t)^2\} \leq 2\ E\{c_t^2\} +\ 2\ E\{s_t^2\} \leq 2\epsilon L_1 + 2C_4\epsilon^2,$$

where we used again (6.46) and Theorem (6.5.1). This last inequality shows that the first term in the right hand side of (6.48) behaves nicely for small ϵ. Finally, the last term behaves nicely as a consequence of Theorem (6.5.1).

Then we completed the proof of **b)**.

By using facts **a)** and **b)** we obtain the result (6.30) stated in the theorem via (6.42).

There remains only to prove that condition (6.36) is satisfied under assumptions (A), (B1) and (C1).
We begin by noticing that

$$E\{\exp[\tfrac{1}{2}\int_{\bar t}^{T}|\psi_t|^2 dt]\} \le \frac{1}{T-\bar t}\int_{\bar t}^{T} E\{\exp[\frac{T-\bar t}{2}|\psi_t|^2]\}dt$$

(this can be proved by using Jensen inequality when we look at functions of time as if they were random variables on the probability space $(\mathcal{T},\mathcal{B}_T,\lambda_T/(T-\bar t))$, where $\mathcal{B}_T$ is the Borel field of $\mathcal{T}$ and λ_T is the Lebesgue measure on $\mathcal{T}$; in this picture the integral w.r.t time is just the expectation). Hence we shall try to find bounds for the integral in the r.h.s. of this last inequality. From the definition of ψ, from the mean value theorem for integrals and from Schwartz inequality we deduce

$$\frac{1}{T-\bar t}\int_{\bar t}^{T} E\{\exp[\frac{T-\bar t}{2}|\psi_t|^2]\}dt \le E\{\exp[(T-\bar t)\zeta_\tau^2]\}\ E\{\exp[(T-\bar t)\frac{s_\tau^2}{\epsilon^2}]\},$$

where τ is a suitable time instant in $\mathcal{T}$. We shall prove that

$$E\{\exp[(T-\bar t)\zeta_t^2]\} < \infty, \quad E\{\exp[(T-\bar t)\frac{s_t^2}{\epsilon^2}]\} < \infty \tag{6.49}$$

for all $t \in \mathcal{T}$. Now we use (6.48), which actually holds independently of (6.36), so that no circularity is present in the proof. Set for simplicity

$$C_5 := 2\frac{C_2}{\pi}k^2\tilde\epsilon, \qquad C_6 := \frac{1+2kC_2}{2C_1},$$

$$C_7^\epsilon := (\frac{C_2}{\epsilon}+C_6)^2, \qquad C_8 := \frac{8}{(b-\tilde\epsilon)}(T-\bar t).$$

The superscript in C_7^ϵ is used to indicate explicitly that this constant depends on ϵ. The first condition in (6.49) reads

$$E\{\exp[C_8(k^2\gamma_t^2 + C_5 + C_7^\epsilon s_t^2 + C_6^2(X_t-\widetilde X_t)^2)]\} < \infty$$

for all t of $\mathcal{T}$. Iterated applications of Schwartz's inequality yield the following set of sufficient conditions for the above inequality to hold:

$$E\{\exp[4C_8k^2\gamma_t^2]\} < \infty, \quad E\{\exp[4C_8C_7^\epsilon s_t^2]\} < \infty, \quad E\{\exp[2C_8C_6^2c_t^2]\} < \infty.$$

By a last application of Schwartz's inequality and noticing that $s_t^2 \leq 2(c_t^2 + \gamma_t^2)$ we obtain the following set of sufficient conditions for (6.49):

$$E\{\exp[\mu_\epsilon c_t^2]\} < \infty, \quad E\{\exp[\mu_\epsilon \gamma_t^2]\} < \infty, \tag{6.50}$$

$$\mu_\epsilon := \max(4C_8k^2, 2C_8C_6^2, 16C_8C_7^\epsilon),$$

for all $t \in \mathcal{T}$.

Now we prove that conditions (6.50) are satisfied. In order to do so, we can reason as in Theorem (4.7) page 137 of [45]. Both the equations (6.31) for c_t and (6.32) for γ_t satisfy conditions (4.139) of page 138 of [45]: Boundedness of the diffusion coefficients is immediate, whereas the drift parts satisfy

$$(f(c_t) - \widetilde{E}\{f\})^2 \leq 2k(\sqrt{\frac{2C_2\epsilon}{\pi}} + c_t^2),$$

and a similar inequality holds for γ_t (these are proven analogously to the proof of Theorem 6.5.1). The proof of the theorem is finally concluded. $\square$

Chapter 7

Stochastic Differential Equations with Finite–Dimensional Density

Hatred is never stilled through hatred in this world; by non-hatred alone is hatred stilled. This is the Eternal Law

Dhammapada I.5

7.1 Introduction

In the present chapter we consider the following problem: Is it possible to maintain the dimension of the density–evolution of a diffusion process stilled?

This chapter can be interesting also to readers who are not too interested in nonlinear filtering. It treats problems related to stochastic differential equations (SDE's) with densities evolving in finite–dimensional exponential families. We consider also the possibility of projecting the density of the solution of a SDE onto a finite–dimensional exponential manifold of densities. Readers interested in filtering will find a way to construct nonlinear–finite–dimensional–exponential optimal filters in the last section of the chapter.

We begin the chapter by solving a first problem: Given a diffusion coefficient and an exponential family, we characterize the SDE's with the given diffusion coefficient whose densities evolve regularly in the given exponential family. This fact leads to the following result: Given an arbitrary diffusion coefficient and an arbitrary exponential family, one can always define a drift such that the density of the resulting diffusion process evolves in the prescribed exponential manifold. In particular, given arbitrary nonlinear diffusion coefficients, one can define drifts in such a way that the resulting densities evolve in a Gaussian manifold. This gives rise to a wide class of nonlinear diffusion processes with Gaussian density.

We then turn to the problem of projecting the Fokker–Planck equation onto an exponential family. Here we shall allow once again a little redundancy so as to keep the chapter as self contained as possible. In particular, notice that a short and informal account on the projection in Fisher metric of the density of a diffusion process onto a finite–dimensional manifold of densities was already given in Chapter 4, Section 4.2. Here we expand that introduction specializing it to exponential families. We saw in Chapter 2 that the solution of the filtering problem is a Stochastic PDE which can be seen as a generalization of the Fokker–Planck equation (FPE) expressing the density of a diffusion process. This equation is called the Kushner–Stratonovich equation (KSE). In Chapter 4, Section 4.4, the Fisher metric was used to project the Kushner–Stratonovich equation onto an exponential manifold of probability densities. This method can be used also for the simpler FPE. In the present chapter we consider the projection in Fisher metric of the density evolution of a diffusion process onto an exponential manifold. Such projection is obtained via the projected FPE. We examine the projected evolution and interpret it as the density evolution of a different diffusion process via the previous result.

We continue by presenting some examples which show how this theory can be used to construct nonlinear SDE's with prescribed (possibly stationary) exponential densities.

An application of the results to mathematical finance is briefly discussed.

Moreover we show that for some particular models convergence of the original density towards an invariant distribution implies existence of a finite–dimensional exponential family for which the projected density converges to the same distribution.

We conclude the chapter with an application to nonlinear filtering. We use the results derived for diffusion processes to derive existence results for filtering problems. Problems concerning finite dimensionality of filters for nonlinear systems have been studied in the past by several authors. In some of these works it was stressed the importance of exponential families. This holds especially for discrete–time systems: See for example Sawitzki (1981) [56]. Runggaldier and Spizzichino (1997) studied finite dimensionality of filters from a Bayesian point of view in [55]. The effects of a non-Gaussian initial condition have been studied by Makowski (1986) [46], and by Sowers and Makowski (1992) [58]. The reader interested in finite dimensionality of filters for nonlinear systems in discrete time can also check [20]. The reference [5] is also of possible interest. Some examples of finite dimensional filters for nonlinear systems are given by Frost (1971) in [26].

Problems on finite dimensionality of nonlinear filters in continuous time have been studied in the past with Lie–algebraic criterions, see for example Hazewinkel, Marcus and Sussmann (1983) [30], Chaleyat–Maurel and Michel (1984) [18], Ocone and Pardoux (1989) [50], Lévine (1991) [44]. In the present section we treat nonlinear filtering problems with discrete time observations, as in Chapter 4 Section 4.5.2. As usual, in order to keep the chapter as self contained as possible, we shall present some facts already given in Chapter 4. Our result shows that given a prescribed (possibly nonlinear) diffusion coefficient for the state equation, a prescribed (possibly nonlinear) observation function and a partially prescribed exponential family, one can define a drift for the state equation such that the resulting nonlinear filtering problem has a solution which is finite dimensional and which stays on the prescribed exponential family.

The material presented in this chapter has partly appeared in Brigo [9]. A related treatment with a deeper geometrical approach can be found in Brigo and Pistone [17]. The application to filtering with continuous time observations is under development.

7.2 Stochastic differential equations and exponential families

In this section we consider the following problem: Given a scalar diffusion coefficient and an exponential manifold of densities, find drifts such that the resulting scalar stochastic differential equations (SDE's) have densities evolving in the prescribed exponential family. This problem has a straightforward solution.

Let us first establish the appropriate framework. On the complete probability space $(\Omega, \mathcal{F}, P)$ let us consider a stochastic process $\{X_t, t \geq 0\}$ of diffusion type. Let the dynamic equation describing X be of the following form

$$dX_t = f_t(X_t)dt + \sigma_t(X_t)dW_t, \quad X_0,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion independent of the initial condition X_0. The equation above is an Itô stochastic differential equation. In the following derivation we treat the scalar case. Consider the following set of assumptions.

(O) The initial state X_0 has a density p_0 w.r.t. the Lebesgue measure on $\mathbf{R}$, and has finite moments of any order. Moreover, $p_0(x) > 0$ for all $x \in \mathbf{R}$.

(A) We make the following assumptions on the coefficients f_t, $a_t := \sigma_t^2$: $f \in C^{1,0}$, $a \in C^{2,0}$ which means that f is once continuously differentiable wrt x and continuous wrt t and a is twice continuously differentiable wrt x and continuous wrt t. We assume also local Lipschitz continuity in x uniformly in t : for all $R > 0$, there exists $K_R > 0$ such that

$$|f_t(x) - f_t(x')| \leq K_R\, |x - x'|,$$

$$\|a_t(x) - a_t(x')\| \leq K_R\, |x - x'|\ ,$$

for all $t \geq 0$, and for all $x, x' \in B_R$, the ball of radius R.

(B) Non–explosion : there exists $K > 0$ such that

$$2x f_t(x) + a_t(x) \leq K\,(1 + |x|^2),$$

for all $t \geq 0$, and for all $x \in \mathbf{R}$.

Under assumptions (O), (A) and (B), there exists a unique solution $\{X_t\,,\, t \geq 0\}$ to the state equation, see Stroock and Varadhan (1979) [59], Theorem 10.2.1 with $\phi(x,t) = x^2$.

Under additional assumptions on the coefficients the density of X_t is absolutely continuous with respect to the Lebesgue measure, and its density $p_t(x)dx := P[X_t \in dx]$ satisfies the Fokker–Planck equation:

$$\begin{aligned}
\frac{\partial p_t}{\partial t} &= \mathcal{L}_t^* p_t, \\
\mathcal{L}_t &= f_t \frac{\partial}{\partial x} + \tfrac{1}{2} a_t \frac{\partial^2}{\partial x^2}, \\
\mathcal{L}_t^* p &= -\frac{\partial}{\partial x}(f_t p) + \tfrac{1}{2} \frac{\partial^2}{\partial x^2}(a_t p).
\end{aligned}$$

Assumptions under which this happens are related to boundedness of the coefficients f, a and of their partial derivatives plus uniform ellipticity of a_t, see [59] Theorem 9.1.9 or [25] Theorem 6.4.7. In order to operate in a functional–space framework, we rewrite the Fokker–Planck equation as an equation in L_2. In order to do so, we need to require $p_t(x) > 0$ for all x, t. This can be obtained by the maximum principle applied to the Fokker-Planck equation in the case of elliptic coefficient a and bounded coefficients described above. Rewrite Fokker–Planck equation for the square root of p_t:

$$\frac{\partial\sqrt{p_t}}{\partial t} = \frac{\mathcal{L}_t^* p_t}{2\sqrt{p_t}}. \tag{7.1}$$

Next, select a finite–dimensional manifold of square roots of exponential densities to approximate $\sqrt{p_t}$. The definition of exponential family is given in Chapter 3, Section 3.3. See also Remark 3.3.2. Remember that in our case n, i.e. the dimension of the state space, amounts to one. In this chapter we shall denote by $EM(c)$ the exponential family whose exponent functions are given by the function $c : \mathbf{R} \to \mathbf{R}^m$. We shall assume the following throughout the chapter:

(C) the exponent functions $c_1, ..., c_m$ have at most polynomial growth and are twice continuously differentiable.

Under this assumption we can always add terms in the exponent to make the family integrable when it is not integrable, obtaining the exponential family $EM(c)$, as in Remark 3.3.2. Consider the set

$$EM(c)^{1/2} = \{\sqrt{p(\cdot,\theta)}, \theta \in \Theta\} \subset L_2$$

of square roots of densities of $EM(c)$. The map $\sqrt{p(\cdot,\theta)} \mapsto \theta$ can be seen as a coordinate system that gives a manifold structure to $EM(c)^{1/2}$ (see Section 3.1 of Chapter 3 for the details). Now consider a generic curve $t \mapsto \sqrt{p(\cdot,\theta_t)}$ on L_2. We consider the following problem.

Problem 7.2.1 *Let be given a diffusion coefficient $a_t(\cdot) := \sigma_t^2(\cdot)$, $t \geq 0$, satisfying assumptions (A) and (B) when $f = 0$. Let be given an exponential family $EM(c)$ satisfying (C). Characterize the SDE's whose initial condition X_0 satisfies (O), and whose densities evolve regularly in the given family $EM(c)$.*

Remark 7.2.2 *Problem 7.2.1 is an extension of a problem of probability and of stochastic processes:*

(i) Which nonlinear functions of Gaussian random variables have a Gaussian distribution?

(ii) Which nonlinear functions of Gaussian processes are Gaussian processes?

These problems are motivated by the filtering problem.

The solution of this problem is given by the following

Theorem 7.2.3 (Solution of Problem 7.2.1) *Assumptions of Problem 7.2.1 in force. Consider the stochastic differential equations*

$$\begin{aligned} dY_t &= u_t(Y_t)dt + \sigma_t(Y_t)dW_t, \quad Y_0 = X_0, \\ u_t(x) &:= \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)\theta_t^T\frac{\partial c}{\partial x}(x) + \\ & -\dot{\theta}_t^T \int_{-\infty}^{x} (c(y) - E_{\theta_t}c)\ \exp[\theta_t^T(c(y) - c(x))]dy, \end{aligned} \tag{7.2}$$

with $t \mapsto \theta_t$ describing C^1–curves in the parameter space Θ.

Then the SDE's (7.2) (considered for all possible regular curves $t \mapsto \theta_t$) solve Problem 7.2.1. Hence the density of Y_t evolves in the prescribed family of exponential densities $EM(c)$.

Proof : Consider an arbitrary (regular) L_2–curve $t \mapsto \sqrt{p(\cdot,\theta_t)}$ evolving in $EM(c)^{1/2}$. Define a diffusion

$$dY_t = u_t(Y_t)dt + \sigma_t(Y_t)dW_t, \quad Y_0 = X_0, \tag{7.3}$$

with the given diffusion coefficient a. We shall define drifts u such that the density of Y_t coincides with the given $p(\cdot,\theta_t)$. Let $\mathcal{T}_t$ be the backward differential operator of Y_t:

$$\mathcal{T}_t = u_t\frac{\partial}{\partial x} + \tfrac{1}{2}a_t\frac{\partial^2}{\partial x^2}.$$

Clearly, the density of Y_t coincides with $p(\cdot,\theta_t)$ if

$$\mathcal{T}^*p(\cdot,\theta_t) = \frac{\partial p(\cdot,\theta_t)}{\partial t},$$

for all $t \geq 0$, which we can rewrite (by the chain rule) as

$$\mathcal{T}^*p(\cdot,\theta_t) = \dot{\theta}_t^T\ [c(\cdot) - E_{\theta_t}c]\ p(\cdot,\theta_t),$$

for all $t \geq 0$. By simple calculations one can rewrite the above equation as the following differential equation for u, where we do not expand the second partial derivative of $a_tp(\cdot,\theta)$:

$$\frac{\partial u_t}{\partial x} + \theta_t^T\frac{\partial c}{\partial x}u_t = \frac{1}{2\,p(\cdot,\theta_t)}\frac{\partial^2}{\partial x^2}(a_tp(\cdot,\theta_t)) - \dot{\theta}_t^T\ [c(\cdot) - E_{\theta_t}c]\ = \mathcal{B}_{t,\theta_t}(\cdot).$$

The solution is unique by standard theory of linear differential equations and is given by

$$u_t(x) := \exp[-\theta_t^Tc(x)]\int_{-\infty}^{x}\mathcal{B}_{t,\theta_t}(y)\exp[\theta_t^Tc(y)]dy,$$

as one can verify immediately by substitution. Straightforward calculations yield

$$u_t(x) = \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)\theta_t^T\frac{\partial c}{\partial x}(x) + \tag{7.4}$$

$$-\dot{\theta}_t^{\,T} \int_{-\infty}^{x} (c(y) - E_{\theta_t}c)\ \exp[\theta_t^T(c(y) - c(x))]dy.$$

□

Example 7.2.4 *An arbitrage–theory interpretation of the solution of Problem 7.2.1*

In this example we apply the result stated in Theorem 7.2.3 to mathematical finance. More precisely, we consider an application to arbitrage theory in continuous time. Suppose we are given a price process $\{B_t, t \in [0\ \ T]\}$ for a risk–free asset and a price process $\{S_t, t \in [0\ \ T]\}$ for a stock, such as

$$dB_t \ = \ r_t B_t dt, \ \ B_0,$$

$$dS_t \ = \ S_t\ f_t(S_t)dt + \ S_t\ \sigma_t(S_t)dW_t, \ \ S_0,$$

and a simple contingent claim $Z = \phi(S_T)$. It is known that the pricing equation, i.e. the PDE which determines the price $\Pi_t(Z) = F(t, S_t)$ at any time t of the derivative related to Z, is given by

$$\partial_t F + r_t\ s\ \ \partial_s F + \tfrac{1}{2}s^2\sigma_t^2\ \ \partial_{ss}^2 F - r_t\ F = 0, \quad F(T,\cdot) = \phi\ ,$$

and *does not depend on* f_t (short term rate of return) but only on σ_t (volatility) (see for example Duffie (1988) [21] for the case of constant coefficients whose generalization is straightforward, or Karatzas (1989) [35]).). This means that the pricing of Z will be based only on the diffusion coefficient (volatility) σ_t so that we can replace f_t as we wish and the pricing of the derivative remains the same. According to Theorem 7.2.3, we can choose $f_t = u_t$ such that the stock price process S has a density evolving in an exponential family $EM(c)$ selected a priori (for example Gaussian). This implies that, as for the pricing aspect, it is not restrictive to assume that a stock-price has an exponential density assigned a priori. The possible implications of this result will be examined in future research work.

Problem 7.2.1 admits a somehow dual problem that one might find interesting.

Problem 7.2.5 *Let be given a finite dimensional class of SDE's. Characterize those families of exponential densities which satisfy the following property:*

There exists a SDE in the given class whose density evolves in the selected exponential family.

If the class of SDE's has the property that all SDE's share the same diffusion coefficient, then the exponential families $EM(c)$ are part of the solution of Problem 7.2.5, but there might be other families solving the problem. This matter will be investigated in future research work.

7.3 Projected density–evolution of a diffusion process

At this point we introduce the geometric structure which permits to project the Fokker–Planck equation onto a finite–dimensional manifold of densities.

As in the previous section, we rewrite the Fokker–Planck equation as an equation in L_2. In order to do so, we require again $p_t(x) > 0$ for all x, t. This can be obtained by the maximum principle applied to the Fokker-Planck equation in the case of elliptic coefficient a and bounded coefficients described in the preceding section. The exponential family $EM(c)$ is also chosen according to the framework of the previous section.

Now consider a generic curve $t \mapsto \sqrt{p(\cdot, \theta_t)}$ on L_2. Its tangent vector in θ_t is given according to the chain rule:

$$\frac{d}{dt}\sqrt{p(\cdot,\theta_t)} = \sum_{i=1}^{m} \frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_i}\,\dot{\theta}_t^{\,i}\ , \tag{7.5}$$

from which we see that tangent vectors in θ_t to all curves lie in the linear (tangent) space

$$\mathrm{span}\{\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_1},\ldots,\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_m}\}.$$

Recall the following quantity

$$g(\theta)_{ij} := 4\langle\frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_i},\frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_j}\rangle,\quad i,j=1,\ldots,m,$$

where $\langle\cdot,\cdot\rangle$ is the inner product of L_2. By straightforward computations,

$$g_{ij}(\theta) = E_\theta\{\frac{\partial\log p(\cdot,\theta)}{\partial\theta_i}\ \frac{\partial\log p(\cdot,\theta)}{\partial\theta_j}\}\ ,\quad i,j=1\ldots m,$$

where $E_\theta\{\phi\} := \int\phi(x)p(x,\theta)dx$, so that $g(\theta)$ is the Fisher information introduced in Chapter 3.1.

Now recall the orthogonal projection defined for all $\theta \in \Theta$ by

$$\Pi_\theta : L_2 \longrightarrow \mathrm{span}\{\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_1},\ldots,\frac{\partial\sqrt{p(\cdot,\theta_t)}}{\partial\theta_m}\} \tag{7.6}$$
$$\Pi_\theta[v] := \sum_{i=1}^m[\sum_{j=1}^m 4g^{ij}(\theta)\,\langle v,\frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_j}\rangle]\;\frac{\partial\sqrt{p(\cdot,\theta)}}{\partial\theta_i}\;.$$

Now we have all the ingredients needed to state the following

Theorem 7.3.1 (Projected density–evolution of an Itô diffusion.) *Consider the assumption*

$$(D) \qquad E_\theta\{\alpha_{t,\theta}^2\} < \infty \quad \forall\theta\in\Theta,\quad \forall t\geq 0,$$

$$\begin{aligned}\alpha_{t,\theta} \quad &:= \quad \frac{\mathcal{L}_t^* p(\cdot,\theta)}{p(\cdot,\theta)} = -f_t\,\frac{\partial}{\partial x}(\theta^T c) - \frac{\partial f_t}{\partial x} + \\ &+ \quad \tfrac{1}{2}[\,a_t\,\frac{\partial^2}{\partial x^2}(\theta^T c) + a_t\,(\frac{\partial}{\partial x}(\theta^T c))^2 + \\ &+ \quad 2\,\frac{\partial a_t}{\partial x}\,\frac{\partial}{\partial x}(\theta^T c) + \frac{\partial^2 a_t}{\partial x^2}\,]\;.\end{aligned}$$

Assume assumptions (O), (A), (B), (C) and (D) on the inital value X_0 and coefficients f, a of the Itô diffusion X and on the exponent functions c of the exponential family $EM(c)$ are satisfied. Assume $p_0(\cdot) = p(\cdot,\theta_0) \in EM(c)$. Then the projection of the Fokker–Planck equation describing the local evolution of $p_t = p_{X_t}$ onto $EM(c)^{1/2}$ reads, in L_2 local coordinates:

$$\frac{\partial}{\partial t}\sqrt{p(\cdot,\theta_t)} = E_{\theta_t}\{\mathcal{L}_t\,c\}^T\;g^{-1}(\theta_t)\;[c(\cdot) - E_{\theta_t}c]\;\;\frac{\sqrt{p(\cdot,\theta_t)}}{2},\quad \sqrt{p(\cdot,\theta_0)} = \sqrt{p_0(\cdot)}$$

and the differential equation describing the local evolution of the parameters for the projected density–evolution is

$$\dot\theta_t = g^{-1}(\theta_t)\;E_{\theta_t}\{\mathcal{L}_t\,c\},\quad \theta_0.$$

Proof : We project the FPE equation (7.1) for $\sqrt{p_t}$ via the projections (7.6), and we obtain the following (m–dimensional) differential equation on the manifold $EM(c)$:

$$\frac{\partial}{\partial t}\sqrt{p(\cdot,\theta_t)} = \Pi_{\theta_t}[\frac{\mathcal{L}_t^* p(\cdot,\theta_t)}{2\sqrt{p(\cdot,\theta_t)}}]. \tag{7.7}$$

Writing the projection map explicitly and comparing with (7.5) yields the following differential equation for the parameters:

$$\dot\theta_t = g^{-1}(\theta_t)\;\int\frac{\mathcal{L}_t^*\,p(x,\theta_t)}{p(x,\theta_t)}\;\frac{\partial p(x,\theta_t)}{\partial\theta}\;dx, \tag{7.8}$$

where integrals of vector functions are meant to be applied to their components.

Notice that in deriving this last equation we did not take in account the specific structure of densities in $EM(c)$. The reasoning given so far holds (but only *formally*) for any finite–dimensional family of densities, provided densities are regular enough. If we ask for this last equation to hold not only in a formal sense, we need to ensure

$$\frac{\mathcal{L}_t^* p(\cdot,\theta_t)}{2\sqrt{p(\cdot,\theta_t)}} \in L_2, \tag{7.9}$$

so that the projection defined in L_2 can act and transform FPE (7.1) on L_2 into equation (7.7) on the manifold $EM(c)^{1/2}$. In the following we consider condition (7.9) when dealing with an exponential family $EM(c)$.

In order to proceed, we keep in mind the results about exponential families given in Lemma 3.3.3. Let us specialize equations (7.7) and (7.8) to $EM(c)$ via Lemma 3.3.3 and let us make sure that they hold not only formally. By using also duality $\mathcal{L}_t - \mathcal{L}_t^*$ we obtain

$$\frac{\partial}{\partial t}\sqrt{p(\cdot,\theta_t)} = \mathcal{P}_{t,\theta_t}\frac{\sqrt{p(\cdot,\theta_t)}}{2}, \tag{7.10}$$

$$\mathcal{P}_{t,\theta_t} := E_{\theta_t}\{\mathcal{L}_t\ c\}^T\ g^{-1}(\theta_t)\ [c(\cdot) - E_{\theta_t}c],$$

and

$$\dot{\theta}_t = g^{-1}(\theta_t)\ E_{\theta_t}\{\mathcal{L}_t\ c\}. \tag{7.11}$$

Equation (7.10) (and consequently equation (7.11)) is well defined as projection of an L_2 equation and admits locally a unique solution if condition (7.9) is satisfied. Condition (7.9) can be rewritten as

$$E_\theta\{\alpha_{t,\theta}^2\} < \infty, \quad \forall\theta\in\Theta, \quad \forall t \geq 0.$$

□

We conclude this section with the following remarks.

Remark 7.3.2 *An example of sufficient conditions under which (D) holds is the following. Assume*

(D1) The functions $f_t,\ \partial_x f_t,\ a_t,\ \partial_x a_t,\ \partial_{xx}^2 a_t,\ \partial_x c,\ \partial_{xx}^2 c$ have at most polynomial growth.

Under this assumption one can add some new functions c in the exponent and obtain a family satisfying (D), in the spirit of Remark 3.3.2

Remark 7.3.3 *Existence of a local solution of the projected equation (7.10) does not require existence of the solution of the original FPE.*

This remark points out that the projected equation may have a solution even when the original FPE has no solution. Actually, consider assumptions (O), (A), (B), (C) and (D1). They ensure existence of the solution of the projected equation, but they do not ensure existence of a solution for the original FPE. For example, f_t can be unbounded, or a_t can be not uniformly elliptic, etc. This means that in some cases our projected density $p(\cdot, \theta_t)$ represents an absolutely continuous – finite–dimensional approximation of the marginal law of X_t which is not absolutely continuous.

7.4 Interpretation of the projected equation

Consider the projected density evolution $p(\cdot, \theta_t)$, expressing the projection in Fisher metric of the density evolution of the one–dimensional diffusion X onto the exponential manifold $EM(c)$. Consider the following problem.

Problem 7.4.1 *Let be given a diffusion coefficient $a_t(\cdot) := \sigma_t^2(\cdot), \quad t \geq 0$ and a drift $f_t(\cdot), \quad t \geq 0$ satisfying assumptions (A) and (B). Consider the SDE*

$$dX_t = f_t(X_t)dt + \sigma_t(X_t)dW_t, \quad X_0,$$

where X_0 satisfies (O). Let be given an exponential family $EM(c)$ satisfying (C). Characterize the SDE's whose initial condition is X_0, whose diffusion coefficient is a, and whose density evolutions coincide with the projected density evolution of X onto $EM(c)$ given by Theorem 7.3.1.

This is just a particular case of Problem 7.2.1, where the curve $t \mapsto \theta_t$ is not arbitrarily chosen but comes from a projection. We can then translate Theorem 7.2.3 for this problem and write the following

Theorem 7.4.2 (Interpretation of the projected density evolution.) *Assume assumptions (O), (A), (B), (C) and (D) on the initial value X_0 and the coefficients f, a of the Itô diffusion X and on the exponent functions c of the exponential family $EM(c)$ are satisfied. Assume $p_0(\cdot) = p(\cdot, \theta_0) \in EM(c)$. Let $p(\cdot, \theta_t)$ be the projected density evolution, according to Theorem 7.3.1. Define*

$$\begin{aligned} dY_t &= u_t^*(Y_t)dt + \sigma_t(Y_t)dW_t, \quad Y_0 = X_0, \\ u_t^*(x) &:= \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)\theta_t^T\frac{\partial c}{\partial x}(x) \\ &\quad -E_{\theta_t}\{\mathcal{L}_t c\}^T g^{-1}(\theta_t)\int_{-\infty}^{x}(c(y) - E_{\theta_t}c)\ \exp[\theta_t^T(c(y) - c(x))]dy. \end{aligned} \tag{7.12}$$

Then Y is the Itô diffusion whose density evolves according to the projected evolution $p(\cdot,\theta_t)$ of X_t onto $EM(c)$.

Proof: Combine Theorem 7.2.3 and Theorem 7.3.1. □

Note that the differential equation for u and its solution can be written in terms of the projection as

$$\frac{\partial u_t}{\partial x} + \theta_t^T \frac{\partial c}{\partial x} u_t = \frac{1}{\sqrt{p(\cdot,\theta_t)}} \{ \frac{1}{2\sqrt{p(\cdot,\theta_t)}} \frac{\partial^2}{\partial x^2}(a_t p(\cdot,\theta_t))$$

$$-\Pi_{\theta_t}[\frac{1}{2\sqrt{p(\cdot,\theta_t)}} \frac{\partial^2}{\partial x^2}(a_t p(\cdot,\theta_t))]$$

$$+\Pi_{\theta_t}[\frac{1}{\sqrt{p(\cdot,\theta_t)}} \frac{\partial (f_t p(\cdot,\theta_t))}{\partial x}]\},$$

and

$$u_t^*(x) = \frac{1}{p(\cdot,\theta_t)} \int_{-\infty}^{x} \{ \frac{1}{2\sqrt{p(y,\theta_t)}} \frac{\partial^2}{\partial x^2}(a_t(y) p(y,\theta_t))$$

$$-\Pi_{\theta_t}[\frac{1}{2\sqrt{p(\cdot,\theta_t)}} \frac{\partial^2}{\partial x^2}(a_t(\cdot) p(\cdot,\theta_t))](y)$$

$$+\Pi_{\theta_t}[\frac{1}{\sqrt{p(\cdot,\theta_t)}} \frac{\partial (f_t(\cdot) p(\cdot,\theta_t))}{\partial x}](y) \;\} \sqrt{p(y,\theta_t)} \; dy$$

Note that the integral appearing in equation (7.12) is well defined under assumption (D) and under the assumption that densities of $EM(c)$ are integrable.

7.5 A simple convergence result

We shall show that in the particular case where we select a *constant* diffusion coefficient, convergence of the density of the original process X implies existence of at least an exponential family such that the projected density $p(\cdot,\theta)$ converges towards the same stationary distribution, no matter how we choose θ_0. Assume then $\sigma_t(x) = 1$ for all x, t. We have the following

Theorem 7.5.1 (Global stability of the projected evolution) *Assume that the diffusion process*

$$dX_t = f(X_t)dt + dW_t, \quad X_0,$$

satisfies assumption (O), (A), (B) with f having at most polynomial growth and nonzero in a set with positive Lebesgue measure. Let F be a primitive of f and *assume that the conditions under which $\bar{p} \propto \exp[2F]$ is the unique stationary density of X are satisfied. Assume*

$$\int f(x)^4 \exp(2F(x))dx < \infty, \quad \int (\partial_x f)^2(x) \exp(2F(x))dx < \infty.$$

Then :

(i) There exists an exponential family $EM(c)$ such that $\bar{p} \in EM(c)$, and

(ii) the projected density $p(\cdot, \theta_t)$ for $EM(c)$ given by Theorem 7.3.1 converges towards $\bar{p}$ for all possible initial $\theta_0 \in \Theta$.

Proof : Set

$$\bar{c}(x) := 2\int_0^x f(z)dz =: 2F(x),$$

and consider the exponential family $EM(\bar{c})$. It is easy to verify that under our assumptions $\bar{c}$ satisfies assumptions (C) and (D). It is a classical result on stationary distributions for diffusion processes that $\bar{p} \propto \exp[1\bar{c}]$ (see for example Kontorovich and Lyandres (1995) [39]).
It is also known that under our assumptions this stationary density is unique. The fact that $\exp[\bar{c}]$ is a stationary distribution can be verified immediately: Straightforward computations yield $\mathcal{L}^* \exp[\bar{c}] = 0$. Consider now the projected density evolution $p(\cdot, \theta_t)$ onto $EM(\bar{c})$. Notice that $\bar{p} = p(\cdot, \bar{\theta})$, $\bar{\theta} = 1$. The ODE describing the evolution of θ_t is, by Theorem 7.3.1,

$$\dot{\theta}_t = g^{-1}(\theta_t)\; E_{\theta_t}[\mathcal{L}_t\; \bar{c}]. \tag{7.13}$$

Notice that

$$E_{\bar{\theta}}[\mathcal{L}_t\; \bar{c}] = \int (\mathcal{L}_t \bar{c})(x) p(x, \bar{\theta}) dx \propto \int \bar{c}(x) \mathcal{L}_t^* \exp(\bar{c}(x)) dx = 0,$$

so that $\bar{\theta}$ is an equilibrium point for the equation of θ_t. Now we prove that this equilibrium is unique and stable. From $\mathcal{L}_t \bar{c} = 2f_t^2 + \partial_x f_t$, by a quick integration by parts we obtain

$$E_\theta[\mathcal{L}_t\; \bar{c}] = 2[\bar{\theta} - \theta] \int f_t(z)^2 p(z, \theta) dz,$$

so that, by examining the right–hand side of (7.13) one sees that (since f is nonzero in a set with positive Lebesgue measure) θ_t strictly increases for $\theta_t < \bar{\theta}$ and strictly decreases for $\theta_t > \bar{\theta}$. Moreover, the equilibrium point $\bar{\theta}$ is clearly unique since the right–hand side of (7.13) is nonzero for any θ different from $\bar{\theta}$. Then $\bar{\theta}$ is the unique globally stable equilibrium point and the proof is complete. $\square$

7.6 Examples

In this section we consider some simple applications of the above theory which lead to nonlinear SDE's whose solutions have Gaussian densities. We shall also consider cases where this Gaussian density is stationary.

7.6.1 The case of a one dimensional zero-mean Gaussian manifold

Consider the one-dimensional exponential manifold $EM(x^2)$. Its densities are given by

$$p(\cdot,\theta) = \sqrt{\frac{-\theta}{\pi}}\exp[\theta x^2],\ \theta < 0.$$

These are Gaussian densities whose mean is zero and whose variance is $-1/(2\theta)$. Take f_t identically zero, and notice that $\mathcal{L}_t c = a_t$. Moreover, the inverse of the Fisher metric is, in this case, $g^{-1}(\theta) = 2\theta^2$. The projected density is described by the solution of the ODE

$$\dot\theta_t = 2\theta_t^2 E_{\theta_t}\{a_t\}.$$

By applying the previously found formula (7.12) to this problem we obtain

$$u_t^\star(x) \quad = \quad \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + [a_t(x) - E_{\theta_t}a_t]x\theta_t.$$

Consider now the particular case of a monomial as diffusion coefficient: $\sigma_t(x) := x^k$, k positive integer. This coefficient does not satisfy assumptions on linear growth ensuring non–explosion. Yet, we proceed anyway and we shall see that the projected parameter reaches the forbidden value $\theta = 0$ in a finite time. This is the same as saying that the varinace in the projected Gaussian density–evolution explodes in a finite time.

After simple computations, one finds the following result: Consider the SDE

$$dY_t = u_t(Y_t)dt + Y_t^k dW_t,$$

$$u_t(y) = ky^{2k-1} + [y^{2k} - \frac{(2k)!!}{(-2\theta_t)^k}]\ y\ \theta_t,$$

$$\theta_t = (-1)^k[\theta_0^{k-1} + (-1)^k(2k)!!(k-1)2^{-k+1}t]^{\frac{1}{k-1}},$$

$$t < \frac{2^{k-1}|\theta_0^{k-1}|}{(k-1)(2k)!!},$$

$$Y_0 \sim \mathcal{N}(0, -\frac{1}{2\theta_0}),\ \theta_0 < 0,$$

where $(2k)!!$ is the product of all integer odd numbers preceding $2k$. This diffusion has Gaussian density

$$Y_t \sim \mathcal{N}(0, -\frac{1}{2\theta_t}).$$

7.6.2 The case of a one dimensional unit-variance Gaussian Manifold

Consider the one-dimensional exponential manifold
$EM(x)$ given by the following densities:

$$p(\cdot, \theta) = \sqrt{\frac{1}{2\pi}} \exp[\theta x - \tfrac{1}{2}x^2 - \tfrac{1}{2}\theta^2], \ \theta \in \mathbf{R}.$$

These are Gaussian densities whose mean is θ and whose variance is 1. Notice that the term $-\frac{1}{2}x^2$ in the exponent defines a reference measure different from the Lebesgue measure. No modification is necessary for the projected Fokker–Planck equation. Notice that $\mathcal{L}_t x = f_t$. Moreover, the inverse of the Fisher metric is $g^{-1}(\theta) = 1$ and does not depend on θ. The projected density is described by the solution of the ODE

$$\dot{\theta}_t = E_{\theta_t}\{f_t\}.$$

By applying the previously found formula (7.12) with slight modifications due to the fixed term $-\frac{1}{2}x^2$ in the exponent of the exponential family we obtain, after straightforward computations:

$$u_t^*(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)[\theta_t - x] + E_{\theta_t}\{f_t\}.$$

Now, according to the choice of f_t, one can obtain different results.

The case $f_t = 0$. Then the projected equation becomes $\dot{\theta}_t = 0$ and hence $\theta_t = \theta_0$ for all $t \geq 0$. The corresponding drift is

$$u_t^*(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)[\theta_0 - x].$$

The SDE

$$dY_t = u_t^*(Y_t)dt + \sigma_t(Y_t)dW_t \quad Y_0 \sim \mathcal{N}(\theta_0, 1),$$

has stationary density $Y_t \sim \mathcal{N}(\theta_0, 1)$ for all possible choices of σ_t. Here the mean in the Gaussian manifold remains constant.

The case $f_t = k$ for all $t \geq 0$. Then the projected equation becomes $\dot{\theta}_t = k$ and hence $\theta_t = \theta_0 + kt$ for all $t \geq 0$. The corresponding drift is

$$u_t^*(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)[\theta_0 + kt - x] + k.$$

The SDE

$$dY_t = u_t^*(Y_t)dt + \sigma_t(Y_t)dW_t \quad Y_0 \sim \mathcal{N}(\theta_0, 1),$$

has density $Y_t \sim \mathcal{N}(\theta_0 + kt, 1)$ for all possible σ_t. Here the mean in the Gaussian manifold evolves linearly in time.

The case $f_t(x) = x$ for all $t \geq 0$ and $x \in \mathbf{R}$. Then the projected equation becomes $\dot{\theta}_t = \theta_t$ and hence $\theta_t = \theta_0 \exp(t)$ for all $t \geq 0$. The corresponding drift is

$$u_t^*(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)[\theta_0 \exp(t) - x] + \theta_0 \exp(t).$$

Then the SDE

$$dY_t = u_t^*(Y_t)dt + \sigma_t(Y_t)dW_t \quad Y_0 \sim \mathcal{N}(\theta_0, 1),$$

has density $Y_t \sim \mathcal{N}(\theta_0 \exp(t), 1)$ for all possible σ_t. Here the mean in the Gaussian manifold evolves exponentially in time.

7.6.3 SDEs with densities evolving in $EM(x^4)$

Consider the exponential family $EM(x^4)$. For this family, we have

$$p(x, \theta) = \rho\ \theta^{1/4}\ \exp(-\theta x^4), \quad \theta > 0, \quad \rho := \frac{2}{\Gamma(1/4)}.$$

In this case $\psi(\theta) = -(\log\theta)/4 - \log\rho$, $E_\theta[x^4] = -1/(4\theta)$, and $g(\theta) = 1/(4\theta^2)$. Let us consider an arbitrary diffusion coefficient a, and take a drift f defined ad hoc according to

$$f_t(x) := -\frac{3a_t(x)}{2x} - \frac{x^5}{5}\ ,$$

such that $E_\theta\{\mathcal{L}_t x^4\} = 4\theta^2$, which causes equation (7.11) to become $\dot{\theta}_t = 1$. As a consequence, $\theta_t = \theta_0 + t$ and, according to Theorem 7.4.2, if we set

$$\begin{aligned} u_t^*(x) \quad &:= \tfrac{1}{2}\frac{\partial a_t(x)}{\partial x} - 2a(x)\ (\theta_0 + t)\ x^3 \\ &- \exp[(\theta_0 + t)x^4] \int_{-\infty}^{x} (-y^4 + \frac{1}{4(\theta_0 + t)}) \exp[-(\theta_0 + t)y^4] dy\ , \end{aligned}$$

we obtain that the SDE

$$dY_t = u_t^*(Y_t)dt + \sigma_t(Y_t)dW_t, \quad Y_0 \sim \rho\ \theta_0^{1/4}\ \exp(-\theta_0 x^4),$$

has density $Y_t \sim \rho\ (\theta_0 + t)^{1/4}\ \exp[-(\theta_0 + t)x^4]$ for all possible σ_t.

7.6.4 SDE's with prescribed diffusion coefficient and with prescribed stationary exponential density

In this section we focus on the following problem. Assume we are given a one dimensional family of exponential densities $EM(c)$, where c is a suitable scalar function. Consider again the diffusion process X with drift f_t and diffusion coefficient σ_t. Suppose we select a drift f_t such that

$$\mathcal{L}_t c = f_t \frac{\partial c}{\partial x} + \tfrac{1}{2} a_t \frac{\partial^2 c}{\partial x^2} = 0 \ \forall t \geq 0.$$

This happens if we take

$$f_t := -\frac{a_t \frac{\partial^2 c}{\partial x^2}}{2 \frac{\partial c}{\partial x}},$$

provided the denominator is non-zero almost everywhere. If this happens, of course

$$\dot{\theta}_t = g^{-1}(\theta_t)\ E_{\theta_t}\{\mathcal{L}_t\ c\} = 0, \ \forall t \geq 0,$$

so that the projected density is

$$p(\cdot, \theta_t) = p(\cdot, \theta_0) \ \forall t \geq 0.$$

The formula for u_t^* specializes to

$$u_t^*(x) := \tfrac{1}{2} \frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2} a_t(x) \theta_0 \frac{\partial c}{\partial x}(x).$$

So we conclude the following: under suitable regularity and growth assumptions on the functions c and σ_t, the SDE

$$dY_t = \tfrac{1}{2} \frac{\partial a_t}{\partial x}(Y_t) dt + \tfrac{1}{2} a_t(Y_t) \theta_0 \frac{\partial c}{\partial x}(Y_t) dt + \sigma_t(Y_t) dW_t,$$

$$p_{Y_0}(x) = \exp[\theta_0 c(x) - \psi(\theta_0)],$$

has stationary density

$$p_{Y_t}(x) = \exp[\theta_0 c(x) - \psi(\theta_0)], \ \forall t \geq 0$$

for all possible σ_t.

7.7 Application to filtering

In this section we present an application to nonlinear filtering. We shall consider the filtering problem with continuous time state and discrete time observations (see Chapter 4, Section 4.5.2). We shall consider an unobserved process X

and a related discrete time measurement process Z. The state and observation equations will be modeled according to

$$\begin{aligned} dX_t &= f_t(X_t)dt + \sigma_t(X_t)\, dW_t, \\ Z_n &= h(X_{t_n}) + V_n. \end{aligned}$$

We shall assume X_0, f and $a_t = \sigma_t^2$ to satisfy assumptions (O), (A) and (B). We shall also assume the following on the observation function h:

(E) h and h^2 are linearly independent. They satisfy assumption (C) and assumption (D) when chosen as c functions (after the possible addition of a new term in the spirit of remark 3.3.2 of Chapter 3).

In this model only discrete–time observations are available, at time instants $0 = t_0 < t_1 < \cdots < t_n < \cdots$, and $\{V_n\,,\, n \geq 0\}$ is a standard Gaussian white noise sequence independent of $\{X_t\,,\, t \geq 0\}$.

The nonlinear filtering problem consists in finding the conditional density $p_n(x)$ of the state X_{t_n} given the observations up to time t_n, i.e. such that $P[X_{t_n} \in dx \mid \mathcal{Z}_n] = p_n(x)\, dx$, where $\mathcal{Z}_n := \sigma(Z_0, \cdots, Z_n)$. We define also the prediction conditional density $p_n^-(x)\, dx = P[X_{t_n} \in dx \mid \mathcal{Z}_{n-1}]$. The sequence $\{p_n\,,\, n \geq 0\}$ satisfies a recurrent equation, and the transition from p_{n-1} to p_n is decomposed in two steps, as explained in [32], [47] :

Prediction step Between time t_{n-1} and t_n, we solve the Fokker–Planck equation

$$\frac{\partial p_t^n}{\partial t} = \mathcal{L}_t^*\, p_t^n \;, \qquad p_{t_{n-1}}^n = p_{n-1} \;.$$

The solution at final time t_n defines the prediction conditional density $p_n^- = p_{t_n}^n$.

Correction step At time t_n, the observation Z_n is combined with the prediction conditional density p_n^- via the Bayes rule

$$p_n(x) \propto \Psi_n(x)\; p_n^-(x) \;, \tag{7.14}$$

modulo a normalizing constant, and $\Psi_n(x)$ denotes the likelihood function for the estimation of X_{t_n} based on the observation Z_n only, i.e.

$$\Psi_n(x) := \exp\left\{ -\tfrac{1}{2}|Z_n - h(x)|^2 \right\} \;. \tag{7.15}$$

Select an exponential family $EM(c^\bullet)$ where $c^\bullet$ satisfies:

(F)

$$c_1^\bullet(x) = h(x),$$

$$c_2^\bullet(x) = h^2(x),$$

and the remaining components of $c^\bullet$ are chosen arbitrarily in such a way that $c^\bullet$ satisfies assumptions (C) and (D) ($c_1^\bullet$ and $c_2^\bullet$ satisfy the assumptions because of assumption (E)).

If we use the exponential family $EM(c^\bullet)$ defined above, then we obtain the projection filter (see Chapter 4) density $p(\cdot,\theta_n)$, and the transition from θ_{n-1} to θ_n is also decomposed in two steps :

Prediction step Between time t_{n-1} and t_n, we solve the ODE coming from the projection of the Fokker–Planck equation:

$$\dot\theta_t^n = g(\theta_t^n)^{-1}\, E_{\theta_t^n}\{\mathcal{L}_t\, c^\bullet\}\ , \qquad \theta_{t_{n-1}}^n = \theta_{n-1}\ .$$

The solution at final time t_n defines the prediction parameters $\theta_n^- = \theta_{t_n}^n$.

Correction step Substituting the approximation $p(\cdot,\theta_n^-)$ into formula (7.14), we observe that the resulting density does not leave the exponential family $EM(c^\bullet)$. Indeed, it follows from (7.15) that

$$\begin{aligned}\Psi_n(x) &= \exp\{\,-\tfrac{1}{2}\,h(x)^2 + h(x)\,Z_n - \tfrac{1}{2}\,Z_n^2\,\}\\ &= \exp\{\,-\tfrac{1}{2}c_2^\bullet(x) + Z_n\,c_1^\bullet(x) - \tfrac{1}{2}\,Z_n^2\,\}\ ,\end{aligned}$$

and the parameters are updated according to the formula

$$\theta_n = \theta_n^- - \begin{bmatrix} -Z_n \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

which is *exact.*

So far we described the projection filter for a given system. Now we plan to construct a filtering problem with the same diffusion coefficient σ in the state, with the same observation function h, the same noises W, V, the same initial condition X_0, and such that its solution stays in the exponential family $EM(c^\bullet)$, which is partly defined by h and partly preassigned.

The method is very simple. We keep on denoting by $p(\cdot,\theta)$ the generic density in $EM(c^\bullet)$ and $g(\theta)$ the related Fisher information matrix. We describe the construction of our filtering problem step by step.

We start with the initial condition X_0, and we assume its density to be in $EM(c^\bullet)$: $p_{X_0} = p(\cdot,\theta_0) \in EM(c^\bullet)$. Define a drift $u^1(\cdot)$ such that the diffusion

$$dY_t = u_t^1(Y_t)dt + \sigma_t(Y_t)dW_t, \quad 0 \le t < t_1, \quad Y_0 = X_0,$$

has density in $EM(c^\bullet)$. This can be done according to Theorem 7.4.2 by defining

$$u_t^1(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)(\theta_t^1)^T\frac{\partial c^\bullet}{\partial x}(x) +$$

$$-E_{\theta_t^1}\{\mathcal{L}_t c^\bullet\}^T g^{-1}(\theta_t^1)\int_{-\infty}^{x}(c^\bullet(y) - E_{\theta_t^1}c^\bullet)\ \exp[(\theta_t^1)^T(c^\bullet(y) - c^\bullet(x))]dy,$$

$$\dot{\theta}_t^1 = g(\theta_t^1)^{-1}E_{\theta_t^1}\{\mathcal{L}_t c^\bullet\}, \quad 0 \le t < t_1, \quad \theta_0^1 = \theta_0.$$

Consider $\theta_1^- := \theta_{t_1}^1$. At time t_1 the first observation $Z_1 = h(Y_{t_1}) + V_1$ is available, and we need to correct our prediction density $p(\cdot,\theta_1^-)$ via Bayes' formula. This corresponds to the following update of our parameter in the family $EM(c^\bullet)$:

$$\theta_1 = \theta_1^- - \begin{bmatrix} -Z_1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The optimal filter at time t_1 for our problem is, by construction,

$$p_{Y_{t_1}|Z_1}(\cdot) = p(\cdot,\theta_1) \in EM(c^\bullet).$$

Now we continue by defining a drift u^2 such that the diffusion

$$dY_t = u_t^2(Y_t)dt + \sigma_t(Y_t)dW_t, \quad t_1 \le t < t_2, \quad Y_{t_1} \sim p(\cdot,\theta_1),$$

has density in $EM(c^\bullet)$. This can be done again by defining

$$u_t^2(x) := \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)(\theta_t^2)^T\frac{\partial c^\bullet}{\partial x}(x) +$$

$$-E_{\theta_t^2}\{\mathcal{L}_t c^\bullet\}^T g^{-1}(\theta_t^2) \int_{-\infty}^x (c^\bullet(y) - E_{\theta_t^2} c^\bullet) \ \exp[(\theta_t^2)^T (c^\bullet(y) - c^\bullet(x))] dy,$$

$$\dot{\theta_t^2} = g(\theta_t^2)^{-1} E_{\theta_t^2}\{\mathcal{L}_t c^\bullet\}, \quad t_1 \le t < t_2, \quad \theta_{t_1}^2 = \theta_1.$$

Consider $\theta_2^- := \theta_{t_2}^2$. At time t_2 the second observation $Z_2 = h(Y_{t_2}) + V_2$ is available, and we need to correct our prediction density $p(\cdot, \theta_2^-)$ via Bayes' formula. This corresponds to the following update of our parameter in the family $EM(c^\bullet)$:

$$\theta_2 = \theta_2^- - \begin{bmatrix} -Z_2 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The optimal filter at time t_2 for our problem is, by construction,

$$p_{Y_{t_2}|Z_2}(\cdot) = p(\cdot, \theta_2) \in EM(c^\bullet).$$

By continuing in the same way, we have proven the following

Theorem 7.7.1 (Nonlinear–finite–dimensional–exponential optimal filters). *Let be given functions f and σ satisfying assumptions (A) and (B). Let $\mathcal{L}_t$ be defined by*

$$\mathcal{L}_t = f_t \frac{\partial}{\partial x} + \tfrac{1}{2} a_t \frac{\partial^2}{\partial x^2}, \quad a_t = \sigma_t^2.$$

Let be given a Brownian motion $\{W_t, \ \ t \ge 0\}$, an observation function h satisfying (E) and let be given a white noise process $\{V_n, \ \ n = 0, 1, \ldots\}$ independent of W. Let be given an exponential family

$$EM(c^\bullet) = \{p(\cdot, \theta) = \exp[\theta^T c^\bullet(\cdot) - \psi(\theta)], \theta \in \Theta\}$$

satisfying (F), and let g be its Fisher information matrix.

Define the stochastic process $\{\theta_t, \ \ t \ge 0\}$ according to

$$\theta_t := \theta_t^n, \quad t_{n-1} \le t < t_n; \qquad \qquad \theta_{t_n} := \theta_n,$$

for all $n \in I\!N$, where θ^n is the solution of the differential equation

$$\dot{\theta_t^n} \ = \ g(\theta_t^n)^{-1} E_{\theta_t^n}\{\mathcal{L}_t c^\bullet\}, \quad t_{n-1} \le t < t_n, \quad \theta_{t_{n-1}}^n = \theta_{n-1},$$

and θ_n is the random variable

$$\theta_n = \theta^n_{t_n} - \begin{bmatrix} -Z_n \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for all $n \in I\!N$. Define

$$\begin{aligned} u_t(x,\theta) \quad := \quad & \tfrac{1}{2}\frac{\partial a_t}{\partial x}(x) + \tfrac{1}{2}a_t(x)\theta^T \frac{\partial c^\bullet}{\partial x}(x) + \\ & -E_\theta\{\mathcal{L}_t c^\bullet\}^T g^{-1}(\theta) \int_{-\infty}^{x} (c^\bullet(y) - E_\theta c^\bullet) \ \exp[\theta^T(c^\bullet(y) - c^\bullet(x))]dy. \end{aligned}$$

Then the filtering problem with state $\{Y_t, \ \ t \geq 0\}$ and observations $\{Z_n, \ \ n = 1, 2, \ldots\}$ given by

$$\begin{aligned} dY_t &= u_t(Y_t, \theta_t)dt + \sigma_t(Y_t)dW_t, \quad Y_0 \sim p(\cdot, \theta_0), \\ Z_n &= h(Y_{t_n}) + V_n, \end{aligned}$$

Y_0 independent of V and W, has the preassigned finite–dimensional exponential solution $p(\cdot, \theta_t) \in EM(c^\bullet)$ for all possible (nonlinear) σ and h satisfying the assumptions above.

Consider the following conjecture: This result could be properly extended to the case of continuous time observations. It seems then, at a first sight, that it contradicts classical results on nonexistence of finite–dimensional filters, such as Chaleyat–Maurel and Michel (1984) [18], Ocone and Pardoux (1989) [50], Lévine (1991) [44]. This contradiction appears a natural consequence of the arbitrariness of σ and h. Nonetheless, there is no real contradiction. Indeed, since $\{\theta_t, \ \ t \geq 0\}$ depends on the observation process Z, the drift itself depends on the past observations. The filtering problem considered above does not satisfy the assumptions of the quoted papers. Indeed, we cannot construct a nonlinear filtering problem with prescribed (nonlinear) σ and h, with drift u *which does not depend on the observation process Z* and whose solution remains finite dimensional. We have to allow for observations-dependent drifts in order to prove our result. However, filtering problems with a drift or diffusion term that depends on the observations naturally arise in stochastic control theory, see [61] and [23]. In such problems the input process, which is measurable on the σ-algebra generated by the observations, enters in the drift and diffusion terms.

Chapter 8

Conclusions and further research

And the Enlightened One uttered this stanza:

'Through many births I sought in vain
The Builder of this House of Pain.
Now, Builder, You are plain to see,
And from this House at last I'm free;
I burst the rafters, roof and wall,
And dwell in the Peace beyond them all.'

Paul Carus, from the Gospel of Buddha.

When he had led them out to the vicinity of Bethany, he lifted up his hands and blessed them. While he was blessing them, he left them and was taken up into heaven.

Luke XXIV.50–51

8.1 Short description of the results

In this section we list shortly the results of the present thesis. The following section will deal with concluding remarks and possible future developments.

The results of this thesis concern the finite–dimensional approximation of distributions obtained via differential–geometric methods and exponential families, and its application to nonlinear filtering.

The filtering problem is difficult and complicated because the optimal filter is *not* finite dimensional in general. The infinite–dimensional stochastic partial–differential equation (SPDE) describing the optimal filter cannot be characterized by a finite set of stochastic differential equations (SDE's). As an alternative to the past remedies (assumed–density filter and extended Kalman filter) based on heuristic considerations, we present a well defined and Geometry–based filter. Our new method to obtain a finite set of SDEs which approximate the infinite–dimensional SPDE for the optimal filter consists of the projection filter (PF). The projection filter is obtained by projecting the SPDE for the optimal filter onto a finite–dimensional manifold of probability densities in Fisher metric.

We use this geometric framework to define and study in detail the projection filter for *exponential families* of probability densities. The advantages of choosing exponential families are:

- Exponential families allow simple filter equations;
- Exponential families give the possibility of defining the *total projection residual*, an L_2 vector whose norm measures the local approximation involved in the projection at each time instant;
- Equivalence between the projection filter for exponential families and the (previously heuristics–based) exponential assumed–density filters;
- A large class of exponential families permits a perfect update step in the filtering algorithm in the case of discrete–time observations.
- Good simulation results for the exponential projection filter applied to the *cubic sensor* problem;
- Results on the nice asymptotic behaviour of the Gaussian projection filter with small observation noise (the Gaussian densities are a particular case of exponential densities);
- Existence of finite–dimensional exponential filters for a class of *nonlinear* systems. Some coefficients of such nonlinear system can be prescribed arbitrarily (provided they are regular enough), and the remaining coefficients can be selected in such a way that the optimal filter evolves in a finite–dimensional exponential family.

The last result comes from results related to existence of stochastic differential equations with prescribed diffusion coefficients whose densities evolve in prescribed exponential families. This result on SDEs leads to a new interpretation of the projection in Fisher metric of the density–evolution of a diffusion process and to the existence result of finite–dimensional optimal filters described above.

8.2 Further research and future developments

The choice of the exponential family

In Chapter 4 we have introduced a new and systematic way of designing approximate finite–dimensional filters.

One major issue left is the choice of the exponential family S. In Section 4.5 we presented a first partial answer to this problem, although with the choice of the family $S_\bullet$ there is still some freedom left in the choice of the dimension m and in the choice of the remaining functions $\{c_{s+1}, \cdots, c_m\}$. This freedom could be used to reduce the total residual norm $r_t^* = r_t^\bullet$ defined in Section 4.5, or it could be used to design an adaptive scheme for the choice of the exponential family S.

Estimating the distance between the optimal filter and the projection filter

It would also be useful to obtain for all $t \geq 0$ an estimate of the distance (Hellinger metric or Kullback–Leibler information) between the optimal–filter density p_t and the projection–filter density p_t^π, in terms of the total residual norm history $\{r_s^*, 0 \leq s \leq t\}$.

Projection filters in discrete time

Finally, we would like to define projection filters for discrete–time systems. We did so by investigating the possible use of projection filters for estimating the volatility of bilateral exchange rates, in the context of applications to mathematical finance. The first results in this direction can be found in [12]. We still have to relate this discrete–time setup with the work of Kulhavý [40], [41]. Another motivation for this study will be to obtain efficient numerical schemes for the solution of the stochastic differential equation satisfied by the projection–filter parameters, i.e. equation (4.12) for a general exponential family S, or equation (4.20) for the family $S_\bullet$.

Further simulations

We hope to be able to perform simulations for systems related to more concrete applications. Although the cubic sensor problem was helpful in studying the projection residual and is considered a good academic example, some simulations for more complicated systems are desirable. It would also be interesting to compute a numerical approximation of the time–evolution of the distance optimal filter–projection filter in some examples.

Small observation noise

It would be interesting to check the effect of enlarging the manifold of densities on the error bound for the mean–square distance between optimal–filter estimate and projection–filter estimate.

Stochastic differential equations with finite dimensional densities

It would be interesting to try and investigate the unsolved characterization problem 7.2.5.

Finite–dimensional optimal filters

The result on existence of finite–dimensional optimal filters for partly–arbitrary nonlinear systems can be extended to the case of continuous time observations. Moreover, further investigation of the possible applications of these results to stochastic control, mathematical finance and stochastic realization theory are planned.

Chapter 9

Summary (English, Dutch, French, Italian)

Let one's thoughts of boundless loving–kindness pervade the whole world, above, below, across, without obscuration, without hatred, without enmity.

Suttanipata 150

So in everything, do to others what you would have them do to you, for this sums up the Law and the Prophets.

Matthew VII.12

Summary (English)

FILTERING BY PROJECTION ON THE MANIFOLD OF EXPONENTIAL DENSITIES

The present thesis treats the finite–dimensional approximation of distributions obtained via differential–geometric methods and exponential families. The key ingredients in the theory developed here are: Stochastic differential equations (SDE's), the filtering problem, the differential geometric approach to statistics, and the theory of exponential families.

SDE's are roughly an extension of ordinary differential equations (ODE's) to the case where the evolution of the system is afflicted by randomness. This

evolution then needs to be described by a mathematical object called SDE, since ODE's do not incorporate randomness.

The filtering problem consists of estimating the state of a stochastic system from noise perturbed observations. One has a system whose state evolves according to a SDE, and one observes a related process which is generally a function of the state process plus some new randomenss. This function is not bijective in general, so that it cannot be inverted to recover the state (even in the case where no new randomness is present in the observations). This is usually referred to as the case of *partial observations*. The filtering problem consists of estimating the signal at any time instant from the history of the observation process up to the same instant.

If the evolution of the state and the observations are described by linear equations, the solution of the problem is the well known Kalman Filter (KF). This filter consists of a *finite set* of recursive equations which permit to update the estimates including at each time instant the new observations. In this case the filter is said to be *finite dimensional*.

The more general nonlinear filtering problem is far more complicated because the resulting nonlinear filter is *not* finite dimensional in general. Finite dimensionality of a filter is loosely defined as a filter consisting of a finite set of recursive equations which update the optimal estimate of the state based on the past observations. In general there is no such set of equations for the nonlinear filtering problem. The solution of the filtering problem in continuous time is the probability distribution of the state given the past and current observations. This solution is described by a mathematical object called a stochastic partial–differential equation. This is in general an infinite–dimensional equation, in the sense that its solution cannot be characterized by the solution of a finite set of (stochastic) differential equations. The past remedies to this infinite dimensionality (assumed–density filter and extended KF) were based on heuristic considerations and not much is known on the quality of their performances.

In this thesis we present a new method to obtain a finite set of SDEs which approximate the infinite–dimensional equation for the optimal filter. We introduce the projection filter (PF), which is a finite–dimensional nonlinear filter based on the differential–geometric approach to statistics. The projection filter is obtained by projecting the infinite–dimensional equation for the optimal filter onto a finite–dimensional manifold. By using geometry, we construct a procedure to project this infinite–dimensional equation onto a finite–dimensional space. This projection is mathematically well defined. Moreover, there is ample choice about what finite–dimensional space one can project upon.

In this thesis we use this geometric framework to define and study in detail the projection filter for *exponential families* of probability densities. In this thesis we present results describing the advantages of choosing exponential families: Simple filter equation, possibility of defining the *total projection residual* measuring the local approximation involved in the projection around each time instant, equivalence with the (previously heuristics–based) assumed–density filters, perfect update step in the case of discrete–time observations. Moreover we present simulation results for the exponential projection filter applied to a particular system called *cubic sensor*. Finally, some results on the nice asymptotic behaviour of the Gaussian projection filter with small observation noise are given. This treats roughly the case where the randomness afflicting the observations becomes small. The Gaussian densities are a particular case of exponential densities.

The framework of an exponential family of densities with parameters described by SDE's and with the differential–geometric structure developed for the filtering problem is useful also for other applications. In the thesis we have solved several problems related to existence of stochastic differential equations. These results are related to areas such as *stochastic realization theory*, mathematical finance, and existence of finite–dimensional optimal filters, as we show in the final chapter.

Samenvatting (Dutch)

FILTERING DOOR MIDDEL VAN PROJECTIE OP DE VARIETEIT VAN EXPONENTIELE DICHTHEDEN

Het onderhavige proefschrift behandelt de eindig–dimensionale benadering van verdelingen verkregen via differentiaal-meetkundige methoden en exponentiële families. De belangrijkste ingrediënten in de hier ontwikkelde theorie zijn: Stochastische differentiaal vergelijkingen (SDV's), het filterprobleem, de differentiaal- meetkundige benadering van de exponentiële families.

SDV's zijn ruwweg gesproken een uitbreiding van de gewone differentiaalvergelijkingen (GDV's) naar het geval waarin de evolutie van het systeem onderhavig is aan stochastische invloeden. Deze evolutie dient dan beschreven te worden door een mathematisch object dat SDV genoemd wordt.

Het filter probleem behelst het schatten van de toestand van een stochastisch systeem uit door ruis aangetaste waarnemingen. Men gaat uit van een systeem waarvan de toestand zich ontwikkelt volgens een SDV, en men observeert een gerelateerd proces dat in het algemeen de som is van een functie van het toestandsproces en een nieuwe kansvariabele. Deze functie is niet bijectief in het algemeen, zodat deze niet geïnverteerd kan worden om de toestand terug te vinden (zelfs in het geval dat er geen sprake is van een nieuwe kansvariabele). Men spreekt hierbij gewoonlijk van het geval van partiële observaties. Het filterprobleem bestaat uit schatting van het signaal op ieder moment, uit de geschiedenis van het waarnemingsproces tot op datzelfde moment.

Als de evolutie van de toestand en de waarnemingen beschreven worden door lineaire vergelijkingen, dan is de oplossing van het probleem gegeven door het bekende Kalman Filter (KF). Het filter bestaat uit een *eindig stelsel* recursieve vergelijkingen waarmee de schattingen bijgesteld kunnen worden op ieder moment aan de hand van de nieuwe waarnemingen. In dit geval noemt men het filter *eindig dimensionaal.*

Het algemenere niet-lineaire filter probleem is veel gecompliceerder omdat het resulterende niet-lineaire filter in het algemeen *niet* eindig dimensionaal is. Eindig-dimensionaliteit van een filter is vrijelijk gedefinieerd als de eigenschap dat een filter beschreven kan worden met behulp van een eindig stelsel vergelijkingen voor die de optimale schatting van de toestand bijwerken gebaseerd op de eerder gedane waarnemingen. In het algemeen is er niet zo'n stelsel vergelijkingen voor het niet-lineaire filterprobleem. De oplossing van het filterprobleem in continue tijd is de kansverdeling van de toestand gegeven de eerdere en actuele waarnemingen. Deze oplossing wordt beschreven door een mathematisch object dat een stochastische partiële–differentiaal vergelijking genoemd wordt. Dit is in het algemeen een oneindig–dimensionale vergelijking, in die zin dat zijn oplossing niet gekarakteriseerd kan worden als de oplossing van een eindige stelsel van (stochastische) differentiaal–vergelijkingen. De eerdere remedies voor deze oneindige dimensionaliteit (aangenomen–dichtheids filter en uitgebreide KF) waren gebaseerd op heuristische overwegingen en er is niet veel bekend over de kwaliteit van hun prestaties.

In dit proefschrift presenteren we een nieuwe methode om een eindig stelsel van SDV-en te verkrijgen die de oneindig-dimensionale vergelijking van het optimale filter benaderen. We introduceren het projectie filter (PF), dat een eindig-dimensionaal niet-lineair filter is gebaseerd op de differentiaal–meetkundige invalshoek tot de statistiek. Het projectie filter wordt verkregen door projectie van de oneindig–dimensionale vergelijking voor het optimale filter op een eindig-dimensionale varieteit. Door meetkunde te gebruiken, construeren we een procedure om deze oneindig-dimensionale vergelijking op een eindig–dimensionale ruimte te projecteren. Deze projectie is wiskundig goed gedefinieerd. Bovendien is er volop keuze voor de eindig-dimensionale ruimte waarop men kan projecteren. In dit proefschrift gebruiken we dit meetkundige raamwerk om het projectie filter in detail te definiëren en te bestuderen voor *expo-*

nentiële families van kansdichtheden. In het proefschrift presenteren we resultaten die de voordelen van de keuze van exponentiële families beschrijven: Een eenvoudige filter vergelijking, de mogelijkheid om een *totaal projectie residu* te definiëren die de locale approximatie meet waarmee de projectie op ieder moment gepaard gaat, equivalentie met de (voorheen op heuristiek gebaseerde) aangenomen dichtheid filters, een prefecte bijwerkings-stap (update-step) in het geval van discrete–tijd observaties. Bovendien presenteren we simulatie resultaten voor het exponentiële projectie filter toegepast op het speciale systeem geheten de *kubische waarnemer* (cubic sensor). Tenslotte worden er enige resultaten gegeven over het prettige asymptotische gedrag van het Gaussische projektie filter met zachte waarnemingsruis (small observation noise). Dit behandelt ruwweg het geval waarbij de de stochastische meetfout klein wordt. De Gaussische dichtheden zijn een speciaal geval van exponentiële dichtheden.

Het raamwerk van een exponentiële familie van dichtheden met parameters beschreven door SDV-en en met de differentiaal–meetkundige structuur ontwikkeld voor het filter probleem is ook bruikbaar voor andere toepassingen. In het proefschrift hebben we verscheidene problemen opgelost gerelateerd aan de existentie van stochastische differentiaal vergelijkingen. Deze resultaten zijn gerelateerd aan gebieden zoals *stochastische realisatietheorie*, mathematische financiering, en existentie van eindig–dimensionale optimale filters, zoals we aantonen in het laatste hoofdstuk.

Résumé (French)

FILTRAGE PAR PROJECTION SUR LA VARIÉTÉ DES DENSITÉS EXPONENTIELLES

Dans cette thése on traite par des méthodes de géométrie différentielle et de familles exponentielles, l'approximation fini dimensionnelle de distributions. Les ingrédients clef de cette théorie sont : les équations différentielles stochastiques (EDS's), le problème du filtrage, l'approche géométrique de la statistique, et la théorie des familles exponentielles.

Les EDS's sont grosso modo une extension des équations différentielles ordinaires (EDO's) dans le cas où l'évolution du système est perturbée de manière aléatoire. Une telle évolution doit être décrite par un objet mathématique appellé EDS, car les EDO n'incluent pas l'aléatoire.

Le problème du filtrage traite l'estimation de l'état d'un système stochastique à partir d'observations bruitées. On a un système dont l'état évolue selon une EDS, et on observe un processus qui est une fonction de l'état plus un nouveau

bruit aléatoire. En général, cette fonction n'est pas bijective, et ne peut donc pas être inversée pour obtenir l'état (meme dans le cas où l'observation n'est pas bruitée). C'est ce que l'on appelle le filtrage avec *observations partielles.* Le probleme du filtrage consiste à estimer le signal à chaque instant, à partir de l'histoire des observations jusqu'à l'instant courrant

Dans le cas ou l'évolution de l'état et des observations est décrite par des équationes linéaires, la solution du problème est donnée par le filtre de Kalman (FK). Ce filtre est donnée par un *ensemble fini* d'équations qui permet de mettre à jour l' estimée courrante en incorporant les nouvelles observations à chaque instant. Dans ce cas, on dit que le filtre est *de dimension finie.*

Le problème plus général du filtrage non linéaire est bien plus compliqué parce que le filtre non linéaire résultant *n'est pas* de dimension finie en général. Grosso modo, un filtre est de dimension finie s'il existe un ensemble fini d'équations récursives qui permettent de mettre à jour l'estimée courrante, en fonction des observations passées. En général il n'existe pas un tel ensemble d'équations pour le problème du filtrage non linéaire. La solution du problème de filtrage en temps continu est la loi de probabilité de l'état conditionnellement aux observations passées et présentes. Cette solution est décrite par un objet mathématique appellé équation aux dérivées partielles stochastique. C'est en général une équation de dimension infinie, ce qui signifie que sa solution ne peut pas être caractérisée par la solution d'un ensemble fini d'équations différentielles (stochastiques). Les solutions utilisées dans le passé (assumed–density filter (ADF) et le filtre de Kalman étendu) etaient basées sur des considérations heuristiques, et on sait peu de chose sur la qualité des estimations.

Dans cette thèse on donne une nouvelle méthode pour obtenir un ensemble fini d'EDS's qui approche l'équation de dimension infinie du filtre optimal. On introduit le filtre par projection (FP), qui est un filtre non linéaire de dimension finie basé sur l'approche géometrique de la statistique. Le filtre par projection est obtenu en projetant l'équation de dimension infinie du filtre optimal sur une variété de dimension finie. En utilisant la géometrie, on construit une procédure pour projeter cette équation de dimension infinie sur un espace de dimension finie. La projection est bien définie d'un point de vue mathématique. En outre, il y a beaucoup de liberté sur le choix de l'espace de dimension finie sur lequel on projette.

Dans cette thèse on utilise la géométrie différentielle pour définir et étudier en détail le FP pour des *familles exponentielles* de densités de probabilité. En outre, on donne des résultats qui justifient l'emploi de familles exponentielles : équation simple pour le filtre, possibilité de définir le *total projection residual* qui mesure l'approximation locale causée par la projection à chaque instant, l'équivalence avec les ADF (qui etaient basés sur des considérations heuristique avant l'introduction du FP), et la mise à jour exacte dans le cas des observations en temps discret. On presente de plus des simulations pour le FP exponentiel appliqué au système appellé *sensor cubique*. Enfin, on donne des resultats sur le bon comportement asymptotique du FP Gaussien avec petit bruit d' observation. On s'intéresse grosso modo au cas ou le bruit d'observation est petit. Les densités gaussiennes sont un cas particulier des densités exponentielles.

La structure d'une famille exponentielle de densité avec paramètre décrit par des EDS et traitée par des méthodes de géometrie différentielle pour le problème de filtrage peut être utilisée aussi pour d' autres applications. Dans cette thèse nous avons resolu beaucoup de problèmes liés à l'existence des EDS. Ces résultats sont reliés à des sujets comme *réalisation stochastique* (stochastic realization theory), les mathématiques financières et l'existence de filtres optimaux de dimension finie, comme nous l'avons montré dans le dernier chapître.

Sunto (Italian)

FILTRAGGIO TRAMITE PROIEZIONE SULLA VARIETÀ DELLE DENSITÀ ESPONENZIALI

Questa tesi tratta l'approssimazione di dimensione finita di distribuzioni, ottenuta attraverso metodi geometrico–differenziali e famiglie esponenziali. Gli ingredienti chiave della teoria sviluppata sono: equazioni differenziali stocastiche (EDS), il problema del filtraggio, l'approccio geometrico–differenziale alla statistica, e la teoria delle famiglie esponenziali.

Le EDS sono grosso modo un'estensione delle equazioni differenziali ordinarie (EDO) al caso in cui l'evoluzione del sistema è perturbata da rumore. Quest'evoluzione deve essere allora descritta da un oggetto matematico chiamato EDS, dato che le EDO non incorporano l'aleatorietà.

Il problema del filtraggio riguarda la stima dello stato di un sistema stocastico, basata su osservazioni perturbate da rumore. Si ha un sistema il cui stato evolve secondo un'EDS, e si osserva un secondo processo che è generalmente una funzione del processo di stato piú una nuova aleatorietà. La funzione non è biettiva in generale, sicché non può essere invertita allo scopo di recuperare lo stato (nemmeno nel caso in cui la nuova aleatorietà non è presente nelle osservazioni). Questo tipo di problema è generalmente detto problema con *osservazioni parziali*. Il problema del filtraggio consiste nello stimare il segnale a ciascun istante sulla base della storia del processo delle osservazioni fino a quell'istante.

Se l'evoluzione dello stato e le osservazioni sono descritte da equazioni lineari, la soluzione del problema è data dal ben noto filtro di Kalman (FK). Tale filtro è formato da un *sistema finito* di equazioni ricorsive che permettono di aggiornare le stime includendo ad ogni istante le nuove osservazioni. In questo caso il filtro è detto *di dimensione finita*.

Il problema piú generale del filtraggio non lineare è molto più complicato perché il filtro risultante *non è* di dimensione finita in genere. Un filtro è detto di dimensione finita quando, in parole povere, può essere descritto da un sistema finito di equazioni ricorsive che aggiornano la stima ottimale dello stato basata sulle osservazioni passate. In generale un tale sistema di equazioni non esiste per problemi di filtraggio non lineari. La soluzione del problema del filtraggio a tempo continuo è un oggetto matematico chiamato equazione differenziale stocastica alle derivate parziali. Si tratta generalmente di un'equazione di dimensione infinita, nel senso che la sua soluzione non può essere caratterizzata dalla soluzione di un insieme finito di equazioni differenziali (stocastiche). I rimedi proposti in passato (Assumed Density Filters (ADF) e filtro di Kalman esteso) sono basati su considerazioni euristiche e poco è noto sulla qualità delle loro prestazioni. In questa tesi presentiamo un nuovo metodo per ottenere un sistema finito di EDS che approssimano l'equazione di dimensione infinita descrivente il filtro ottimale. Introduciamo il filtro proiezione (FP). Il FP è un filtro non lineare di dimensione finita basato sull'approccio geometrico–differenziale alla statistica. Il filtro proiezione si ottiene proiettando l'equazione di dimensione infinita del filtro ottimale su una varietà di dimensione finita. Usando la geometria, costruiamo una procedura per proiettare questa equazione di dimensione infinita su uno spazio di dimensione finita. Tale proiezione è matematicamente ben definita. Inoltre, si ha ampia scelta sullo spazio di dimensione finita sul quale proiettare. Nella tesi usiamo questa impostazione geometrica per definire e studiare in dettaglio il FP per *famiglie esponenziali* di densità. Presentiamo risultati che descrivono i vantaggi della scelta di famiglie esponenziali: equazioni semplici per il filtro, la possibilità di definire il *total projection residual* per misurare l'approssmazione locale che si ha ad ogni istante, l'equivalenza con l' ADF (che era precedentemente basato su considerazioni euristiche), aggiornamento esatto nel caso di osservazioni in tempo discreto. Inoltre presentiamo simulazioni per il FP esponenziale applicato al particolare sistema noto come *sensore cubico*. Infine, presentiamo alcuni risultati sul comportamento asintotico del FP Gaussiano con piccolo rumore nelle osservazioni. Questo riguarda, a grandi linee, il caso in cui l'aleatorietà che disturba le osservazioni diviene piccola. Le densità Gaussiane sono un caso particolare delle densità esponenziali.

L'impostazione data da una famiglia esponenziale di densità con parametri descritti da EDS e con una struttura geometrico–differenziale, utilizzata per il problema del filtraggio, si dimostra utile anche in altri campi. In questa tesi risolviamo alcuni problemi collegati all' esistenza di equazioni differenziali stocastiche. Questi risultati sono connessi ad aree quali *realizzazione stocastica* (stochastic realization theory), finanza matematica, ed esistenza di filtri ottimali di dimensione finita, come mostriamo nel capitolo finale.

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