Statistical Analysis of k-Nearest Neighbor Collaborative Recommendation

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Outline

Collaborative recommendation

- Collaborative recommendation is a Web information-filtering technique that typically
  - gathers information about your personal interests
  - compares your profile to other users with similar tastes
  - and then gives personalized recommendations.
- Examples include recommending books, people, restaurants, movies, CDs and news.
- Websites such as amazon.com, match.com, movielens.org and allmusic.com already have recommendation systems in operation.

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Users and items

- Collaborative systems deal with two types of variables: users and items.
- The problem: Estimate ratings for items that have not yet been consumed by a user.
- The recommendation process typically starts by asking users a series of questions.
- Personal ratings are then collected in a matrix.

Based on this prior information, the recommendation engine must be able to automatically furnish ratings of as-yet unrated items.

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Example

	Rocky	Platoon	Rambo	Rio Bravo	Star wars	Titanic
Jim	NA	6	7	8	9	NA
James	3	NA	10	NA	5	7
Steve	7	NA	1	NA	6	NA
Mary	NA	7	1	NA	5	6
John	NA	7	NA	NA	3	1
Lucy	3	10	2	7	NA	4
Stan	NA	7	NA	NA	1	NA
Johanna	4	5	NA	8	3	9

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Lucy	3	10	2	7	NA	4
Stan	NA	7	NA	NA	1	NA
Johanna	4	5	NA	8	3	9
Bob	NA	3	3	4	5	?

State of the art

- A number of practical methods have been proposed, including
  - ▷ machine learning-oriented techniques (e.g., Abernethy et al., 2009)
  - ▷ statistical approaches (e.g., Sarwar et al., 2001)
  - ▷ and numerous other ad hoc rules (Adomavicius and Tuzhilin, 2005).
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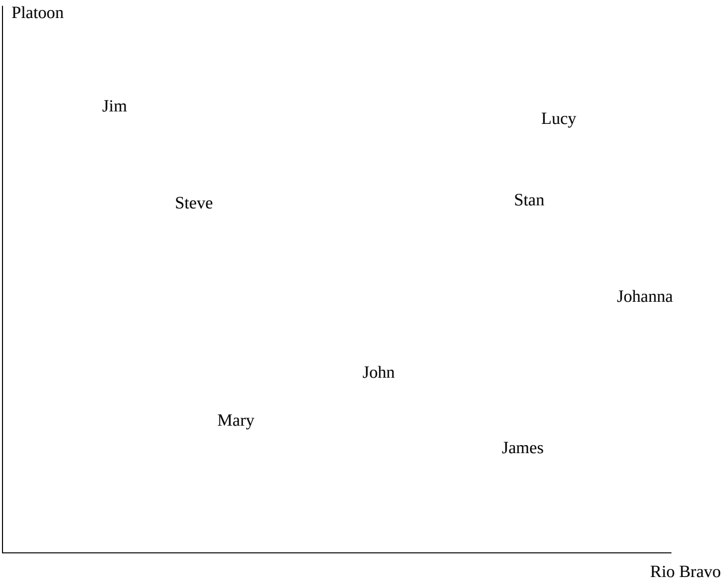
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Lucy
Steve
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Johanna
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Mary
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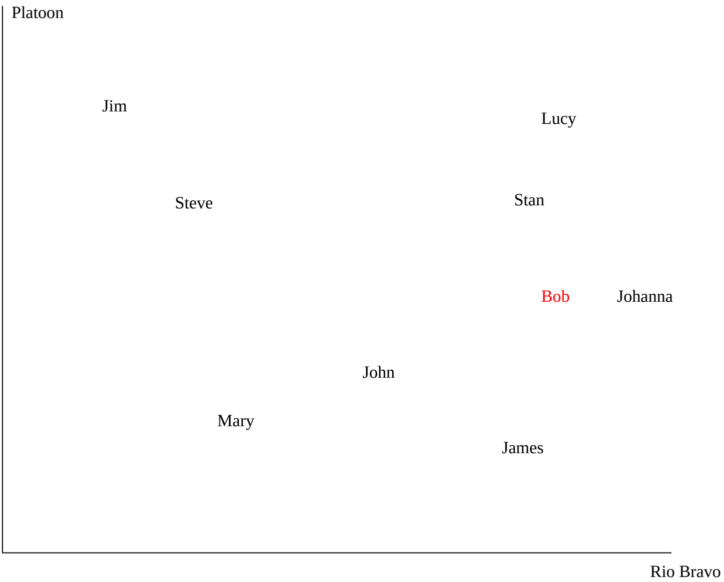
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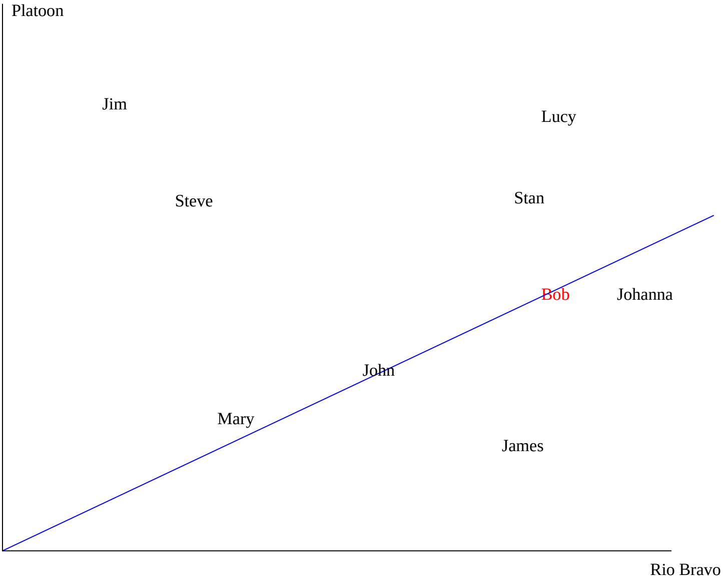
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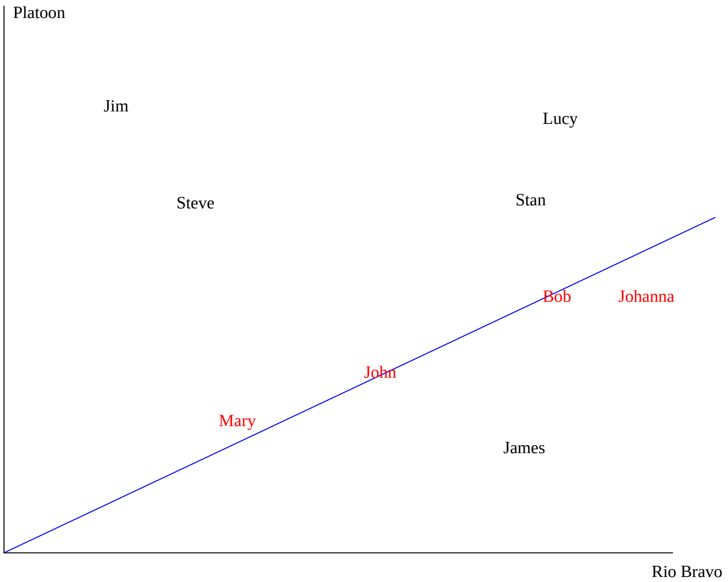
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Our mission

- Despite wide-ranging literature, very little is known about the statistical properties of recommendation systems.
- In fact, no clear probabilistic model even exists.
- To provide an initial contribution to this, we propose
  - to set out a general stochastic model for collaborative recommendation
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Ratings matrix and new users

- Suppose that there are $d+1$ possible items, n users in the ratings matrix.
- Users' ratings take values in the set $(\{0\} \cup [1, s])^{d+1}$.
- A new user Bob reveals some of his preferences for the first time, rating some of the first d items but not the $(d+1)$th.

We want to design a strategy to predict Bob's rating of Titanic, using

1. Bob's ratings of some of the other d movies and
2. the ratings matrix.

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A model for the new user

- We model the preferences of Bob by a random vector $(\mathbf{X}, Y)$ taking values in the set $[1, s]^d \times [1, s]$.
- In fact, we do not observe the variable $\mathbf{X}$, but instead some "masked" version of it $\mathbf{X}^\star$.
- The random variable $\mathbf{X}^\star = (X_1^\star, \ldots, X_d^\star)$ is naturally defined by

$$X_j^\star = \begin{cases} X_j & \text{if } j \in M \\ 0 & \text{otherwise,} \end{cases}$$

where M stands for some non-empty random subset of $\{1, \ldots, d\}$.

Bob	NA	3	3	4	5	?

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A first model for the ratings matrix

- The preferences of users already in the database will be represented by independent random pairs $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$ from the distribution $(\mathbf{X}, Y)$.
- A first idea for dealing with potential non-responses is to consider in place of $\mathbf{X}_i$ its masked version $\widetilde{\mathbf{X}}_i = (\widetilde{X}_{i1}, \ldots, \widetilde{X}_{id})$ defined by

$$\widetilde{X}_{ij} = \begin{cases} X_{ij} & \text{if } j \in M_i \cap M \\ 0 & \text{otherwise.} \end{cases}$$

Drawback

As time goes by, each user in the database may reveal online more and more preferences.

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A more pertinent model

- At time 1, there is only one user in the database, and the subset of items he decides to rate is modeled by a random variable M_1^1.
- At time 2, a new user enters the game and reveals his preferences according to a variable M_2^1, with the same distribution as M_1^1.
- At the same time, user 1 may update his list of preferences, modeled by a random variable M_1^2 satisfying $M_1^1 \subset M_1^2$.
- And so on...

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Preference updating

	Time 1	Time 2	...	Time i	...	Time n
User 1	M_1^1	M_1^2	...	M_1^i	...	M_1^n
User 2		M_2^1	...	M_2^{i-1}	...	M_2^{n-1}
$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$
User i				M_i^1	...	M_i^{n+1-i}
$\vdots$					$\ddots$	$\vdots$
User n						M_n^1

Assumptions

1. The distribution of $(M_i^n)_{n\geq 1}$ is independent of i. It is therefore distributed as a generic random sequence $(M^n)_{n\geq 1}$, satisfying $M^1 \neq \emptyset$ and $M^n \subset M^{n+1}$ for all $n \geq 1$.

2. There exists a random integer n_0 such that $M^{n_0} = \{1, \ldots, d\}$.

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The sequential model

- We let the masked version $\mathbf{X}_i^{(n)}$ of $\mathbf{X}_i$ be defined as

$$X_{ij}^{(n)} = \begin{cases} X_{ij} & \text{if } j \in M_i^{n+1-i} \cap M \\ 0 & \text{otherwise.} \end{cases}$$

- We denote by $(\mathcal{R}_n)_{n\geq 1}$ the subset of users who have already provided information about Titanic at time n. It satisfies $\mathcal{R}_n \subset \mathcal{R}_{n+1}$.

The statistical problem

Estimate the regression function $\eta(\mathbf{x}^\star) = \mathbb{E}[Y|\mathbf{X}^\star = \mathbf{x}^\star]$, based on the database observations $(\mathbf{X}_1^{(n)}, Y_1), \ldots, (\mathbf{X}_n^{(n)}, Y_n)$.

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The cosine-type k_n-NN regression method

- The method estimate $\eta(\mathbf{x}^\star)$ by a local averaging over those Y_i for which
  1. $\mathbf{X}_i^{(n)}$ is "close" to $\mathbf{x}^\star$ and
  2. $i \in \mathcal{R}_n$, that is, we effectively "see" the rating Y_i.
- The closeness between users is assessed by a cosine-type similarity, defined by

$$\bar{S}(\mathbf{x}, \mathbf{x}') = \frac{\sum_{j \in \mathcal{J}} x_j x'_j}{\sqrt{\sum_{j \in \mathcal{J}} x_j^2} \sqrt{\sum_{j \in \mathcal{J}} x_j'^2}},$$

  where $\mathcal{J} = \{j \in \{1, \ldots, d\} : x_j \neq 0 \text{ and } x'_j \neq 0\}$.
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  2. $i \in \mathcal{R}_n$, that is, we effectively "see" the rating Y_i.
- The closeness between users is assessed by a cosine-type similarity, defined by

$$\bar{S}(\mathbf{x}, \mathbf{x}') = \frac{\sum_{j \in \mathcal{J}} x_j x_j'}{\sqrt{\sum_{j \in \mathcal{J}} x_j^2} \sqrt{\sum_{j \in \mathcal{J}} x_j'^2}},$$

  where $\mathcal{J} = \{j \in \{1, \ldots, d\} : x_j \neq 0 \text{ and } x_j' \neq 0\}$.
- If $\mathcal{J} = \{1, \ldots, d\}$ then $\bar{S}(\mathbf{x}, \mathbf{x}') = \cos(\mathbf{x}, \mathbf{x}')$.

The cosine-type k_n-NN regression method

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Example

$$\bar{S}(\text{Bob}, \text{Jim}) = \bar{S}((0, 3, 3, 4, 5), (0, 6, 7, 8, 9)) \approx 0.99,$$

whereas

$$\bar{S}(\text{Bob}, \text{Lucy}) = \bar{S}((0, 3, 3, 4, 5), (3, 10, 2, 7, 0)) \approx 0.89.$$

A key observation

If $M \subset M_i^{n+1-i}$, the positive real number y which maximizes the similarity between $(\mathbf{x}^\star, y)$ and $(\mathbf{X}_i^\star, Y_i)$ is given by

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The estimate

- This suggests the regression estimate

$$\eta_n(\mathbf{x}^\star) = \|\mathbf{x}^\star\| \sum_{i \in \mathcal{R}_n} W_{ni}(\mathbf{x}^\star) \frac{Y_i}{\|\mathbf{X}_i^{(n)}\|},$$

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- The weights are computed according to the penalized similarity

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The regression function

Theorem

Suppose that $\eta_n(\mathbf{X}^\star) \to \eta(\mathbf{X}^\star)$ *in probability as* $n \to \infty$. *Then*

$$\eta(\mathbf{X}^\star) = \|\mathbf{X}^\star\| \, \mathbb{E}\left[\frac{Y}{\|\mathbf{X}^\star\|} \Big| \frac{\mathbf{X}^\star}{\|\mathbf{X}^\star\|} \right] \quad a.s.$$

Fundamental requirement

The regression function $\eta(\mathbf{x}^\star)$ has the special form

$$\eta(\mathbf{x}^\star) = \|\mathbf{x}^\star\| \varphi(\mathbf{x}^\star), \quad \text{where} \quad \varphi(\mathbf{x}^\star) = \mathbb{E}\left[\frac{Y}{\|\mathbf{X}^\star\|} \Big| \frac{\mathbf{X}^\star}{\|\mathbf{X}^\star\|} = \frac{\mathbf{x}^\star}{\|\mathbf{x}^\star\|} \right].$$

In particular, if $\tilde{\mathbf{x}}^\star = \lambda \mathbf{x}^\star$ with $\lambda > 0$, then $\eta(\tilde{\mathbf{x}}^\star) = \lambda \eta(\mathbf{x}^\star)$.

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Suppose that $k_n \to \infty$, $|\mathcal{R}_n| \to \infty$ *a.s. and* $\mathbb{E}[k_n/|\mathcal{R}_n|] \to 0$ *as* $n \to \infty$. *Then*

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Thus, to achieve consistency, the number of nearest neighbors k_n should

- tend to infinity
- but be small with respect to the users who have already rated the item of interest.

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- Consider, to start with, the somewhat ideal situation where all users in the database have rated the item of interest.
- In this case, $\mathcal{R}_n = \{1, \ldots, n\}$, and the asymptotic conditions on k_n become $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$.
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Example 2

- Fix, for simplicity, $\mathcal{R}_1 = \{1\}$.
- At step $n \geq 2$, we decide or not to add one element to $\mathcal{R}_{n-1}$ with probability $p \in (0,1)$, independently of the data.
  1. $\mathcal{R}_n$ is increased by picking a random variable B_n uniformly over the set $\{1, \ldots, n\} - \mathcal{R}_{n-1}$, and set $\mathcal{R}_n = \mathcal{R}_{n-1} \cup \{B_n\}$.
  2. Otherwise, $\mathcal{R}_n = \mathcal{R}_{n-1}$.
- Clearly, $|\mathcal{R}_n| - 1$ has binomial $\mathcal{B}(n-1, p)$ distribution.
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Fact 1

Recall that, for a fixed $i \in \mathcal{R}_n$, the random variable $\mathbf{X}_i^\star = (X_{i1}^\star, \ldots, X_{id}^\star)$ is defined by

$$X_{ij}^\star = \begin{cases} X_{ij} & \text{if } j \in M \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{X}_i^{(n)} = \mathbf{X}_i^\star$ as soon as $M \subset M_i^{n+1-i}$. For each $i \in \mathcal{R}_n$,

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where d is the usual Euclidean distance on $\mathbb{R}^d$.

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Let, for all $i \geq 1$,

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- We write

$$\mathbb{E}\left|\sum_{i\in\mathcal{R}_n} W_{ni}(\mathbf{X}^\star)\frac{Y_i}{\|\mathbf{X}_i^{(n)}\|}-\varphi(\mathbf{X}^\star)\right|$$
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Regularity assumption

The function

$$\varphi(\mathbf{x}^\star) = \mathbb{E}\left[\frac{Y}{\|\mathbf{X}^\star\|}\middle|\frac{\mathbf{X}^\star}{\|\mathbf{X}^\star\|} = \frac{\mathbf{x}^\star}{\|\mathbf{x}^\star\|}\right]$$

satisfies a Lipschitz-type property with respect to the similarity $\bar{S}$. That is, there exists a constant $C > 0$ such that, for all $\mathbf{x}$ and $\mathbf{x}'$ in $\mathbb{R}^d$,

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| \leq C\sqrt{1 - \bar{S}(\mathbf{x}, \mathbf{x}')}.$$

In particular, for $\mathbf{x}$ and $\mathbf{x}' \in \mathbb{R}^d - \mathbf{0}$ with the same null components, this property can be rewritten as

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Rates of convergence

Theorem

Let $\alpha_{ni} = \mathbb{P}(M^{n+1-i} \not\ni M \,|\, M)$. *Then there exists* $C > 0$ *such that, for all* $n \geq 1$,

$$\mathbb{E}\,|\eta_n(\mathbf{X}^\star) - \eta(\mathbf{X}^\star)| \leq C \left\{ \mathbb{E}\left[\frac{k_n}{|\mathcal{R}_n|}\sum_{i \in \mathcal{R}_n} \mathbb{E}\,\alpha_{ni}\right] + \mathbb{E}\left[\prod_{i \in \mathcal{R}_n} \alpha_{ni}\right] + \mathbb{E}\left[\left(\frac{k_n}{|\mathcal{R}_n|}\right)^{P_n}\right] + \frac{1}{\sqrt{k_n}} \right\},$$

where $P_n = 1/(|M| - 1)$ *if* $k_n \leq |\mathcal{R}_n|$, *and* $P_n = 1$ *otherwise.*

- The performance improves as the α_{ni} decrease, i.e., as the proportion of rated items growths.
- The term $\mathbb{E}[(k_n/|\mathcal{R}_n|)^{P_n}]$ can be interpreted as a bias term in dimension $|M| - 1$, whereas $1/\sqrt{k_n}$ represents a variance term.

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Example 1

- In this ideal model, $\mathcal{R}_n = \{1, \ldots, n\}$.
- Suppose in addition that $M = \{1, \ldots, d\}$.
- Then the upper bound becomes

$$\mathbb{E}\left|\eta_n(\mathbf{X}^\star) - \eta(\mathbf{X}^\star)\right| = \mathrm{O}\left(\left(\frac{k_n}{n}\right)^{1/(d-1)} + \frac{1}{\sqrt{k_n}}\right).$$

- There is no influence of the dynamical rating process.
- In particular, the choice $k_n \sim n^{2/(d+1)}$ leads to

$$\mathbb{E}\left|\eta_n(\mathbf{X}^\star) - \eta(\mathbf{X}^\star)\right| = \mathrm{O}\left(n^{-1/(d+1)}\right).$$

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Example 2

- At time 1, the user rates exactly 4 items by randomly guessing in $\{1, \ldots, d\}$.
- At time 2, he updates his preferences by adding exactly one rating among his unrated items, randomly chosen in $\{1, \ldots, d\} - M_1^1$.
- Similarly, at time 3, the user revises his preferences according to a new item uniformly selected in $\{1, \ldots, d\} - M_1^2$, and so on.
- In such a scenario, $M^j = \{1, \ldots, d\}$ for $j \geq d - 3$. Calculations show that

$$\alpha_{ni} = \begin{cases} 0 & \text{if } i \leq n - d + 4 \\ 1 - \dfrac{\binom{d-4}{n-i}}{\binom{d}{n+4-i}} & \text{if } n - d + 5 \leq i \leq n. \end{cases}$$

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- We obtain
$$\mathbb{E}\left[\prod_{i\in\mathcal{R}_n} \alpha_{ni}\right] \leq \frac{C}{n}.$$

- Also,
$$\mathbb{E}\left[\left(\frac{k_n}{|\mathcal{R}_n|}\right)^{P_n}\right] \leq C\left(\frac{k_n}{n}\right)^{1/3}.$$

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