THE NONLINEAR FILTERING PROBLEM

$$\begin{cases} x_k = F_k(x_{k-1}) + w_k \\ y_k = H_k(x_k) + v_k \end{cases}$$

Dynamical equation. n-dimentional

Measurement equation . m-dimentional

Estimation of the conditional law :

$$p_{k/k}(x_k / y_1, \ldots, y_k) = p_k(x_k / y_1, \ldots, y_k)$$

Hyp : the noises are white Gaussian with resp. cov. S_k, R_k

THE NONLINEAR FILTERING PROBLEM

Given $p_{0/0}(x_0)$

Compute the predictive density :

$$p_{k/k-1}(x_k / y_1,\ldots,y_{k-1}) = \int_{R^n} \mathcal{N}(x_k - F_k(u)/S_k)\, p_{k-1}(du / y_1,\ldots,y_{k-1})$$

Compute the corrected density :

$$p_k(x_k / y_1,\ldots,y_k) = \frac{p_{k/k-1}(x_k / y_1,\ldots,y_{k-1})\,\mathcal{N}(y_k - H_k(x_k)/R_k)}{\int_{R^n} p_{k/k-1}(x_k / y_1,\ldots,y_{k-1})\,\mathcal{N}(y_k - H_k(x_k)/R_k)\,dx_k}$$

Where $\mathcal{N}(x/Q) = \exp(-\frac{1}{2}x^T Q^{-1} x)/\sqrt{|2\pi Q|}$

Grid method costly (n high), Monte Carlo methods, ok

THE PARTICLE FILTER

Basic PF

Given $(X_0^1,\ldots,X_0^N)$ $w_0^i \equiv 1/N$

Compute the predictive density :

$$p_{k/k-1}(x_k / y_1,\ldots,y_{k-1}) = \sum_{i=1}^{N} w^i_{k/k-1}\,\delta(x_k = X^i_{k/k-1}) \qquad (X^i_{k/k-1} = F(X^i_k) + W^i)$$

Compute the corrected density :

$$p_k(x_k / y_1,\ldots,y_k) = \sum_{i=1}^{N} w^i_k\,\delta(x_k = X^i_{k/k-1}) \qquad (w^i_k \propto w^i_{k/k-1}\exp[y_k - H_k(X^i_{k/k-1})/R_k])$$

likelihood

Resampling necessary to avoid some degenaracy problems

Filtre particulaire : Analyse de l'erreur locale

$$\square(\square) = \frac{q(y/\square)}{\int q(y/\square)\,p(\square)\,d\square}\,p(\square) = w(\square)\,p(\square)$$

Posterior — Likelihood — Prior

HYP $(\square_1,\ldots,\square_N) \sim p(\square)$ BUT : estimer $\square = E_{\square}[\square(\square)]$

SIR : $\hat{\square} = \sum_{i=1}^{N} w_i \square(\square_i)$

As. non biaisé

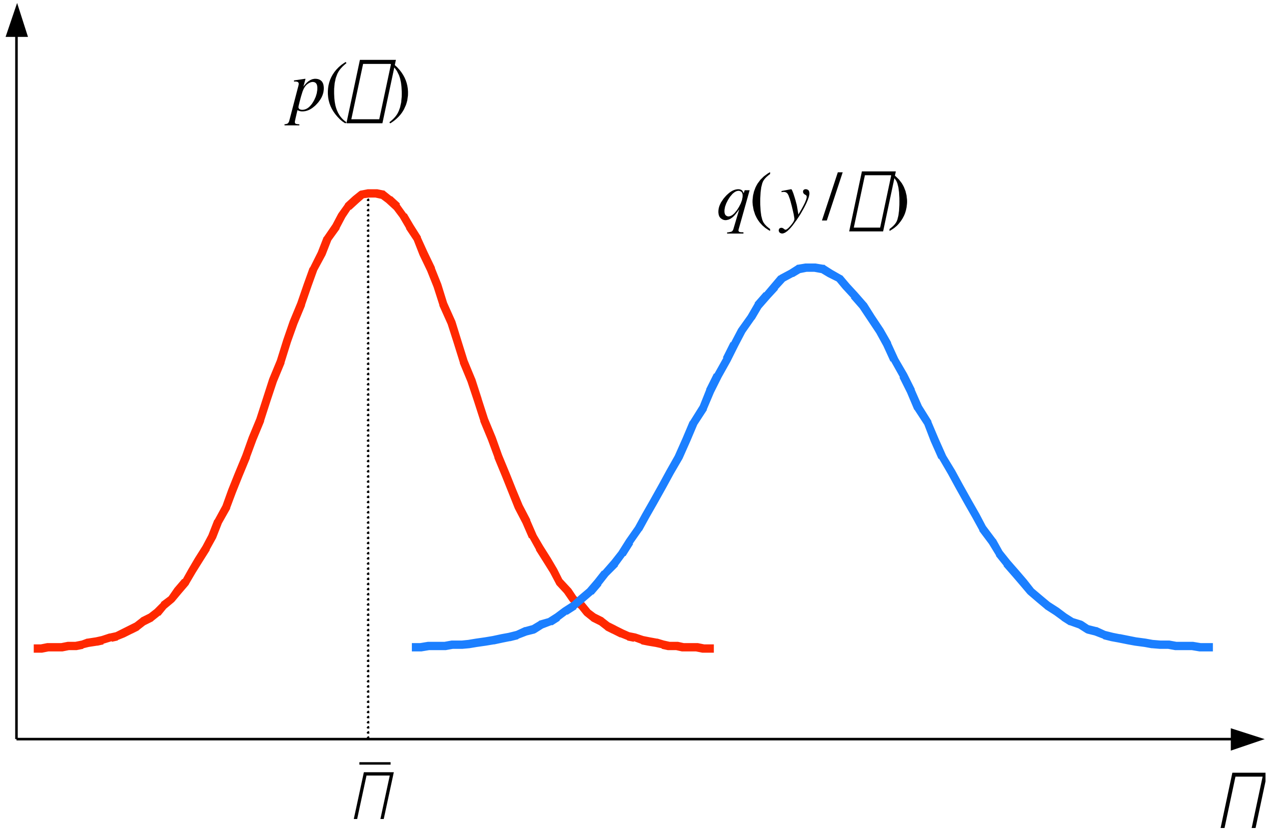
Variance de $\hat{\square}$?

Rejection $\hat{\square} = \frac{1}{N}\sum_{i=1}^{N} \square(\square_i^*)$

$$\begin{cases} \square \sim p(\square) \\ Si\ q(y/\square) \geq cU\ alors\ \square_i^* = \square \end{cases} \qquad \begin{pmatrix} c \geq \sup_{\square} q(z/\square) \\ U \sim unif.[0\ 1] \end{pmatrix}$$

Coût de l'algo (Pa) ?

Analyse de l'erreur locale : cohérence prior/measurement



$$p(\theta) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{1}{2\sigma_2^2}(\theta - \bar{\theta})^2\right) \qquad q(z/\theta) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1}{2\sigma_1^2}(z - h(\theta))^2\right)$$

Analyse de l'erreur locale : cohérence prior/measurement

$$\mathrm{var}(\hat{\theta}) \approx \frac{1}{N}\frac{(\sigma_1^2 + h'^2\sigma_2^2)}{\sigma_1(\sigma_1^2 + h'^2\sigma_2^2)^{1/2}}\exp[C(z-\bar{h})^2]$$

$$C = \frac{\sigma_2^2 h'^2}{(\sigma_1^2 + h'^2\sigma_2^2)(\sigma_1^2 + 2h'^2\sigma_2^2)}$$

$$\bar{h} = h(\bar{\theta}) \quad h' = h'(\bar{\theta})$$

$$\sigma_1\,(likelihood) \quad \sigma_2(prior)$$

- error ↗ with $(z-\bar{h})$
- h' ⟶ 0 : pas d'information -propagation de la densité (var prior)
- h' ↗ : information (PCRB) - var ↗

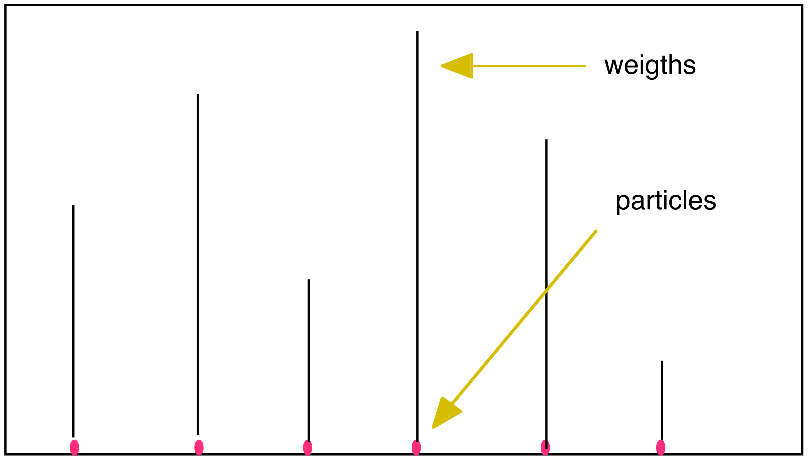
$$P_a = \frac{\sigma_1}{(\sigma_1^2 + h'^2\sigma_2^2)^{1/2}}\exp[-\frac{(z-\bar{h})^2}{2(\sigma_1^2 + h'^2\sigma_2^2)}]$$

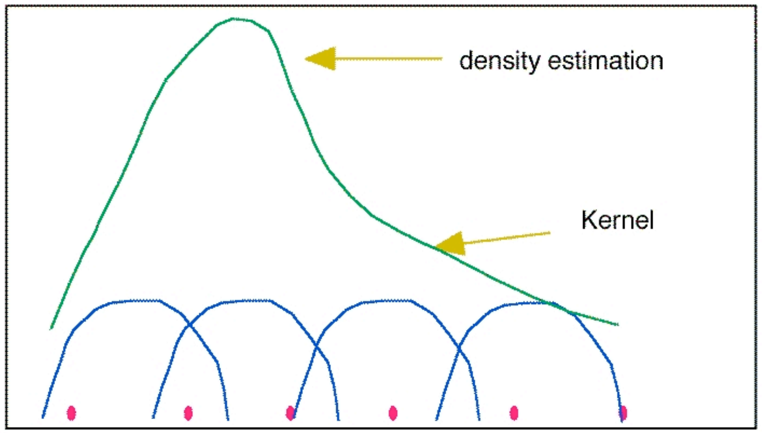
Analyse duale

Améliorations

- Regularized Particle Filter (RPF)
- Choix de la proposal density
- Rao-Blackwelllisation Particle Filter (RBPF)
- Méthodes hybrides : Kalman local, mixture de gaussiennes

RPF





$$\hat{f}(x) = \sum_{i=1}^{N} w^i \delta_{x = X^{(j)}}$$

$$\hat{f}(x) \approx \sum_{i=1}^{N} w^i K[h^{-1} A^{-1} (x - X^{(j)})]$$

Utile pour les bruits de dynamique faibles

RBPF

Décomposition de l'état : $x_k = (x_k^P, x_k^K)$

Sachant x_k^P la dynamique et la mesure sont linéaires

$$\begin{cases} x_{k+1}^P = f^P(x_k^P) + F_k^P(x_k^P)x_k^K + w_k^P \\ x_{k+1}^K = f^K(x_k^P) + F_k^K(x_k^P)x_k^K + w_k^K \end{cases}$$

$$y_k = h(x_k^P) + H(x_k^K) + v_k$$

$$p(X_k^P, x_k^K / Y_k) = p(x_k^K / X_k^P, Y_k)\, p(y_k / X_k^P, Y_{k-1})\, p(x_k^P / X_{k-1}^P Y_{k-1})\, p(X_{k-1}^P / Y_{k-1})$$

Gaussiennes

Dimension des particules + faible

The Kalman-Particle Kernel Filter (KPKF)

Framework : Kernel decomposition (RPF), smooth distribution, case Gaussian kernel

$$p_k(x_k / y_1,\ldots,y_k) = \sum_{i=1}^{N} w_k^i \square(x_k - X^i_{k/k-1} / h^2 P_{k/k-1})$$

- Bandwidth (small smoothing factor) : $h^{n+4} = \square \dfrac{4}{N(n+2)}$

- Sample weighted covariance matrix : $P_{k/k-1} = \operatorname{cov}(X^i_{k/k-1} / w_k)$

We try to preserve this mixture representation

The Kalman-Particle Kernel Filter (KPKF)

Initialisation $$p_{1/0}(x_1) = \sum_{i=1}^{N} w^i_{1/0} \Phi(x_1 - X^i_{1/0} / P^i_{1/0})$$

$$P^i_{1/0} = h^2 \operatorname{cov}(X^i_{1/0} / w_{1/0}) \quad w^i_{1/0} = 1/N$$

Suppose now we have at time k a predicted sample $(X^i_{k/k-1}, P^i_{k/k-1})\ i = 1,\ldots,N$

small

The Kalman-Particle Kernel Filter (KPKF)

Correction

$$p_k(x_k / y_1,\ldots,y_k) = \sum_{i=1}^{N} w^i_{k/k-1} \mathcal{N}(x_k - X^i_{k/k-1} / P^i_{k/k-1}) \mathcal{N}(y_k - H_k(x_k) / R_k)$$

≈ 0 *if* x_k *not close to* $X^i_{k/k-1}$

Linearization around $X^i_{k/k-1}$: $\quad y_k - H_k(x_k) \approx y_k - y^i_{k/k-1} + H^i_k(x_k - X^i_{k/k-1})$

$$\begin{cases} y^i_{k/k-1} = H_k(X^i_{k/k-1}) \\ H^i_k = \nabla H_k(X^i_{k/k-1}) \end{cases}$$

The Kalman-Particle Kernel Filter (KPKF)

Correction

$$p_k(x_k / y_1,\ldots,y_k) = \sum_{i=1}^{N} w^i_{k/k-1}\,\mathcal{N}(x_k - X^i_{k/k-1} / P^i_{k/k-1})\,\mathcal{N}(y_k - y^i_{k/k-1} + H^i_k(x_k - X^i_{k/k-1}) / R_k)$$

Linear wrt x_k : Kalman correction

$$\begin{cases} X^i_k = X^i_{k/k-1} + G^i_k(y_k - y^i_{k/k-1}) \\ P^i_k = P^i_{k/k-1} - P^i_{k/k-1} H^{iT}_k (\Sigma^i_k)^{-1} H^i_k P^i_{k/k-1} \end{cases}$$

$$\begin{cases} \Sigma^i_k = H^i_k P^i_{k/k-1} H^{iT}_k + R_k \\ G^i_k = P^i_{k/k-1} H^{iT}_k (\Sigma^i_k)^{-1} \end{cases}$$

Remain small (of order h^2)

The Kalman-Particle Kernel Filter (KPKF)

Correction : mixture representation preserved

$$p_k(x_k / y_1, \ldots, y_k) = \sum_{i=1}^{N} w_k^i \, \phi(x_k - X_k^i / P_k^i)$$

small

$$w_k^i \propto w_{k-1}^i \, \phi(y_k - y_{k/k-1}^i / \Sigma_k^i)$$

The Kalman-Particle Kernel Filter (KPKF)

Prediction

$$p_{k+1/k}(x_{k+1}/y_1,\ldots,y_k)=\sum_{i=1}^{N} w_k^i \int_{R^n} \Phi(x_{k+1}-F_{k+1}(u)/S_{k+1})\underbrace{\Phi(u-X_k^i/P_k^i)}_{\approx\, 0 \text{ if } u \text{ not close to } X_k^i}du$$

Linearization of $F_{k+1}(u)$ around X_k^i

$$p_{k+1/k}(x_{k+1}/y_1,\ldots,y_k)=\sum_{i=1}^{N} w_k^i \Phi(x_{k+1}-F_{k+1}(X_k^i)/\underbrace{F_{k+1}^i P_k^i F_{k+1}^{iT}+S_{k+1}^i}_{\text{No more small}})$$

We want to avoid total resampling

The Kalman-Particle Kernel Filter (KPKF)

Resampling

Partial :

Approximate $p_{k+1/k}(x_{k+1}/y_1,\dots,y_k) = \sum_{i=1}^{N} w_k^i \square(x_{k+1} - F_{k+1}(X_k^i)/P^i_{k+1/k})$

by the mixture $\hat{p}_{k+1/k}(x_{k+1}/y_1,\dots,y_k) = \sum_{i=1}^{N} w_k^i \square(x_{k+1} - X^i_{k+1/k}/P_{k+1/k})$

Small ($P_{k+1/k}$)

Criterion : MISE (Mean Integrated Square Error)

Total : $(w_k^i \equiv 1/N)$ if the weights are far from uniform

$$Entropy = \sum_{i=1}^{N} w_k^i \log(w_k^i) \geq Threshold$$

The Kalman-Particle Kernel Filter (KPKF)

Partial resampling : principle

$$If\ \xi^i \sim \tilde{p}(x) = \sum_{i=1}^{N} w^i \Phi(x - X^i / P^i - \tilde{h}^2 P) \quad then$$

$$\hat{p}(x) = \sum_{i=1}^{N} w^i \Phi(x - \xi^i / h^2 P)$$

Is a good approximation, with $(h, \tilde{h})$ to be optimizated (balance bias/variance) of :

$$p(x) = \sum_{i=1}^{N} w^i \Phi(x - X^i / P^i)$$

Constrains : $P^i - \tilde{h}^2 P \geq 0 \Rightarrow \tilde{h} \leq \min[eigen\ value\ (C^{-1} P^i (C^T)^{-1})] \quad where \quad CC^T = P$

The Kalman-Particle Kernel Filter (KPKF)

Summary of the algorithm

- Entropy $\leq$ threshold
  - $P^{i}_{k+1/k}$ small : mixture (correction/prediction) with collection of EKF
  - $P^{i}_{k+1/k}$ big : partial resampling
- Entropy > threshold : total resampling

POSTERIOR CRAMER-RAO BOUND (PCRB)

$$J_{ij} = E_{X,Y}\left[-\frac{\partial^2 \log p_{X,Y}(X,Y)}{\partial X_i \partial X_j}\right]$$

Information matrix (Fisher)

$$E\{(g(Y)-X)(g(Y)-X)^T\} \geq J^{-1} = PCRB$$

Estimator of X

Cov matrix

Formule récursive

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POSTERIOR CRAMER-RAO BOUND (PCRB)

Cas d'une dynamique linéaire

$$\begin{cases} X_{k+1} = F_k X_k + W_k \\ y_k = H_k(X_k) + V_k \end{cases}$$

$$J_{k+1} = E_{X_{k+1}}\{[\nabla_{X_{k+1}} H_k^T(X_{k+1})] R_k^{-1} [\nabla_{X_{k+1}} H_k^T(X_{k+1})]^T\} + (F_k J_k^{-1} F_k^T + S_k)^{-1}$$

MC évaluation — Gain du à la variation variation de h — Perte due à la dynamique

- Généralisation de l'équation de Riccati
- Formulation informationnelle du Kalman si h linéaire

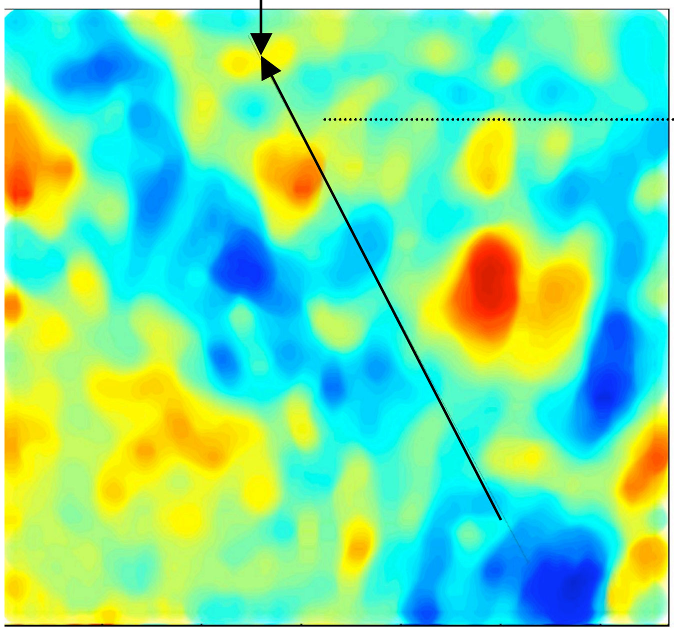
POSTERIOR CRAMER-RAO BOUND (PCRB)

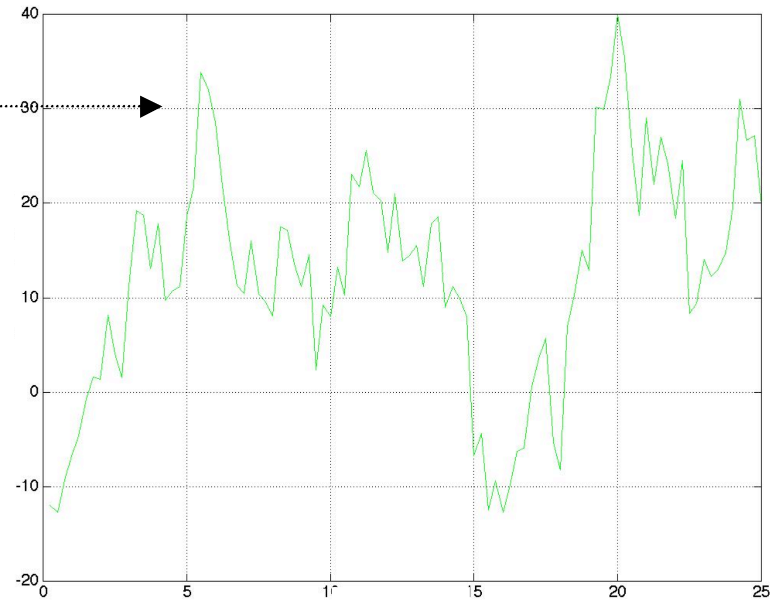
- Evaluation des performances du filtre et du comportement
- Evaluation du systeme/measurement
- Donne des régions de confiances (hyp monomodale à long terme)
- Facile à implémenter

Il est + facile d'estimer l'erreur attendue sur un paramètre que d'estimer sa moyenne avec les méthodes Monte Carlo…

FP - Pondération des trajectoires
Poids des particules

Trajectoire vraie



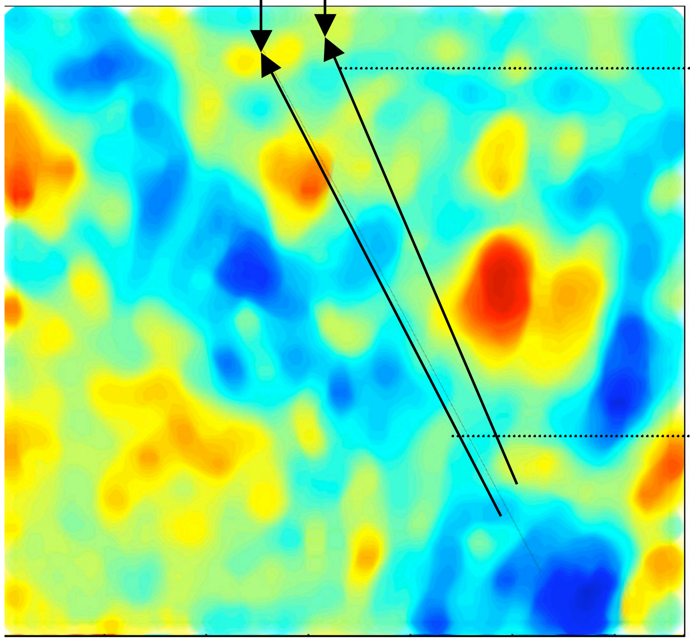


Carte de la grandeur physique

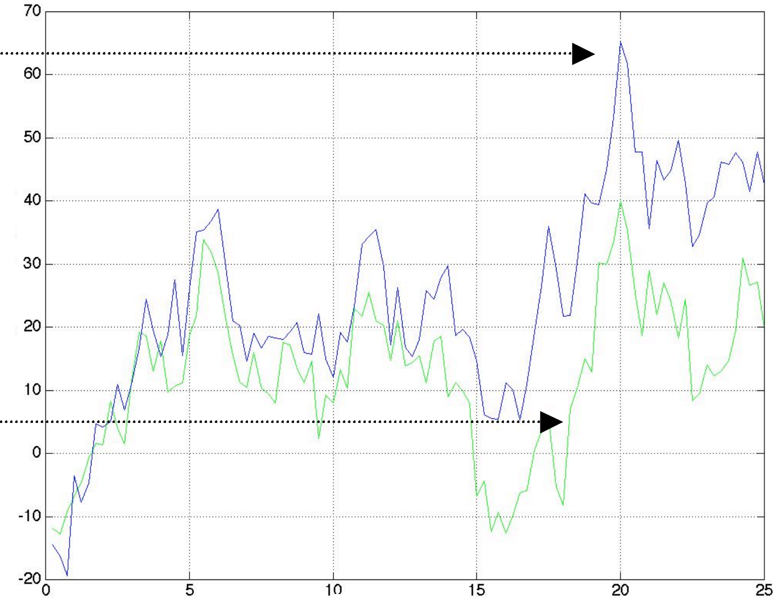
Relevé de la grandeur physique

FP - Pondération des trajectoires
Poids des particules

Trajectoire vraie   Trajectoire candidate



Carte de la grandeur physique

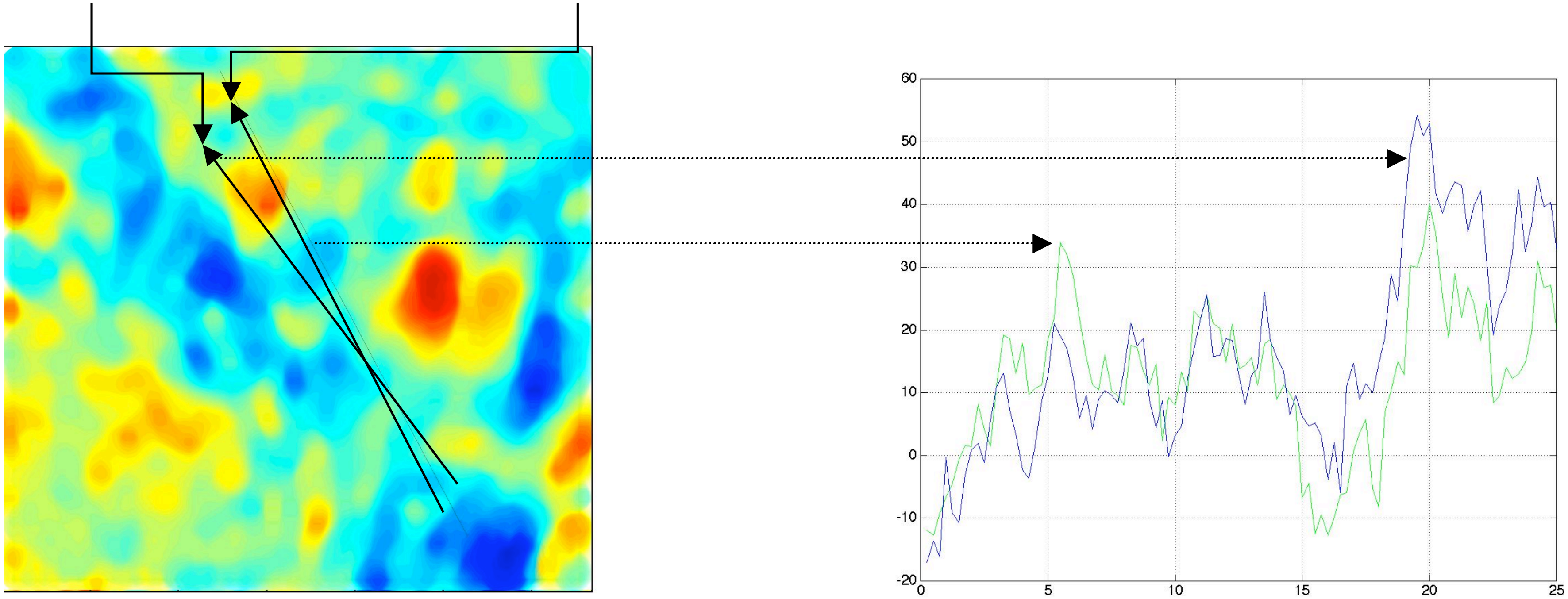


Relevé de la grandeur physique

Assez bonne adéquation :
pondération moyenne de la particule

FP - Pondération des trajectoires
Poids des particules

Trajectoire candidate Trajectoire vraie

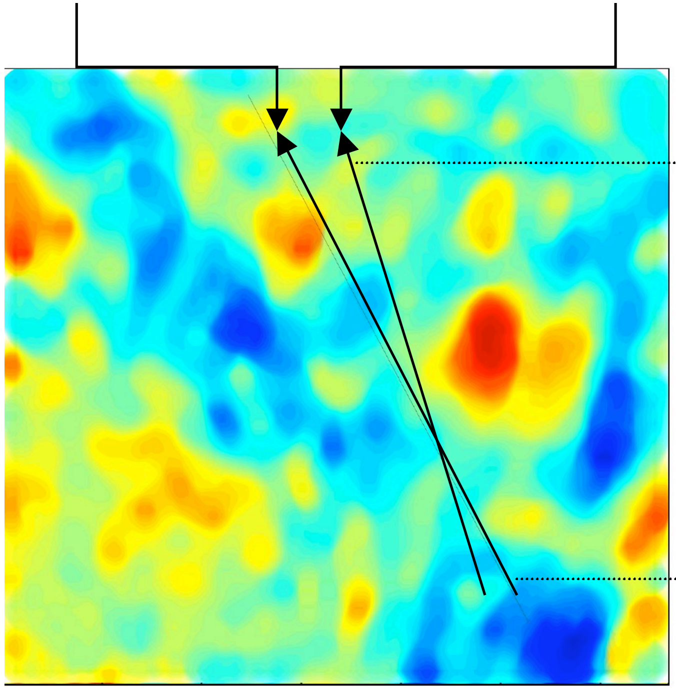


Carte de la grandeur physique

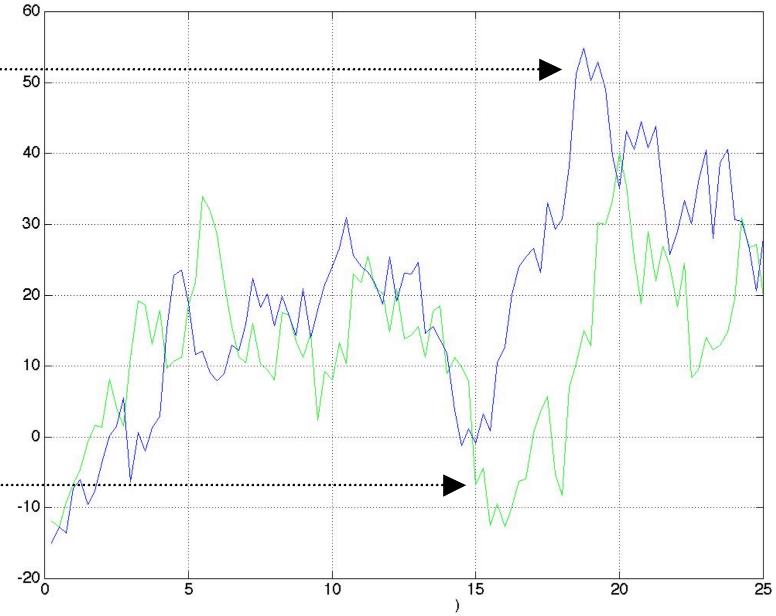
Relevé de la grandeur physique

Bonne adéquation :
pondération forte de la particule

FP - Pondération des trajectoires
Poids des particules



Carte de la grandeur physique

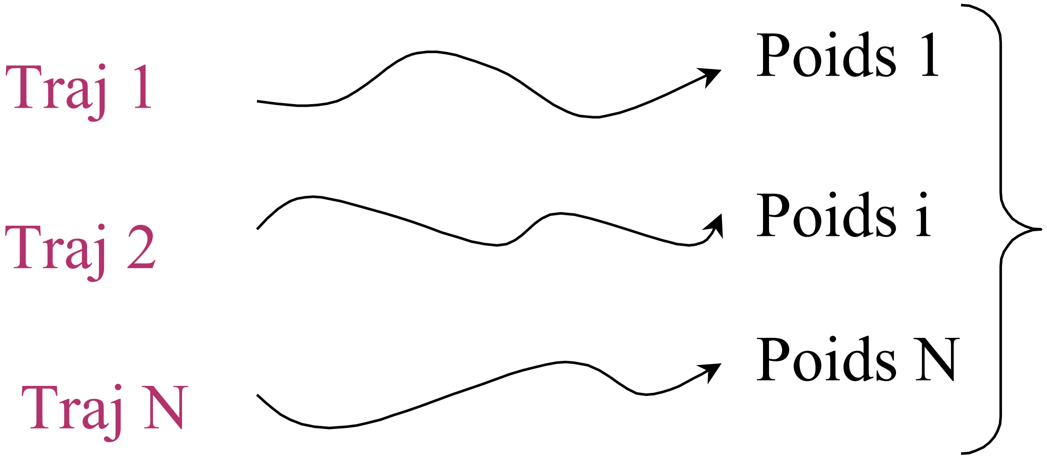


Relevé de la grandeur physique

Mauvaise adéquation :
pondération faible de la particule

FP : Estimation de la position/vitesse

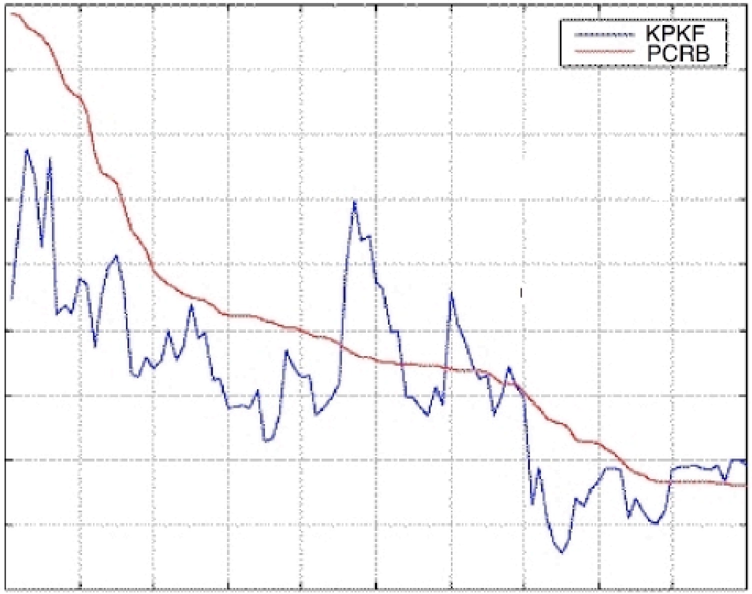
Pilotage séquentiel d’un échantillon de particules.
Une particule est une trajectoire candidate, virtuelle.
Exploration aléatoire de l’espace.

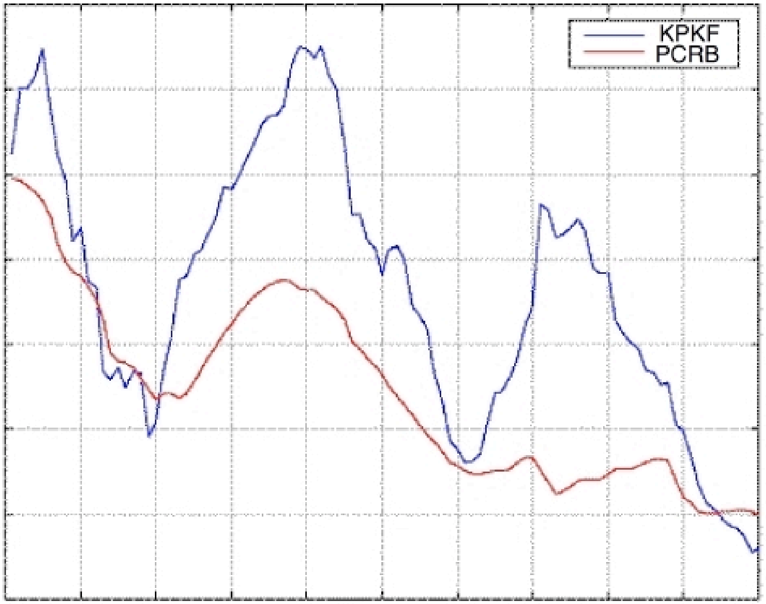


Estimation de la position/vitesse
par moyenne pondérée

(Typiquement avec 10000 particules)

Performance du Filtre KPKF
Estimation des positions





3 % de divergences

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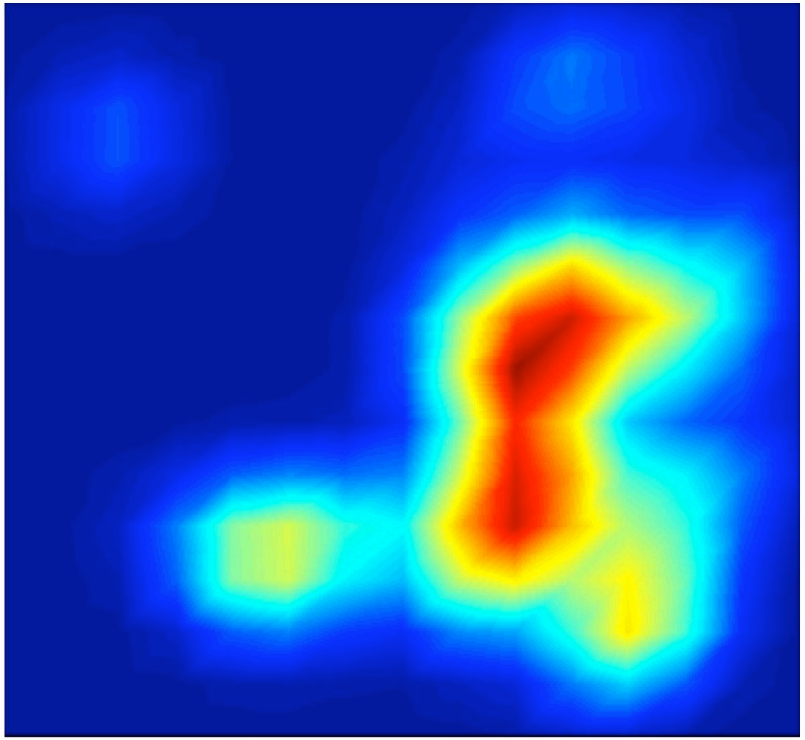
Coupe 2D : densité conditionnelle



CONCLUSIONS

Quand utiliser le FP ?

- Méthodes usuelles inopérantes (MV, Kalman,…)
- Densité assez lisse
- Calcul online ou offline ?

Précautions

- Calcul de la PCRB
- Divergences : superviser le filtre
- Algorithmes adaptés au problème

REFERENCES

RPF
Christian Musso, Nadia Oudjane, François Le Gland
chapitre 3.7 (« Analysis and implementation issues of regularised particle filters ») du livre intitulé « Sequential Monte-Carlo methods in practice ». Chapman & Hall 2001.

PROPOSAL DENSITY
A. Doucet, S.J. Godsill, C. Andrieu, « On sequential Simulation based methods for Bayesian filtering ». Statistics and Computing, vol. 10, no. 3, pp. 197-208, 2000.

RBPF
N. Bergman, L. Ljung, F. Gustafsson. « Terrain navigation using Bayesian statistics ». IEEE Control System Magazine, 19(3), pp. 33-40, 1999.

KPKF
D. T. Pham, K.Dahia, C. Musso.
« A Kalman-Particle Kernel Filter and its Application to Terrain Navigation». Congrès fusion2003 à Cairns (Australie). Juillet 2003

PCRB
P. Tichavsky, C.H. Muravchik, A. Nehorai. « Posterior Cramer-Rao Bounds for discrete-time nonlinear filtering ». IEEE, Vol 46, N° 5, pp. 1386-1396, May, 1998.

Méthodes de resampling partiel bientôt téléchargeables

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