On the Brownian directed polymer in a Gaussian random environment

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1) Definition of the model

(i) The polymer

- $\{\omega_t;\ t \geq 0\}$ d-dimensional Brownian motion
- Defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$
- Probability, Expected values: P_ω^x, E_ω^x

(ii) The environment

- Gaussian landscape B on $\mathbb{R}_+ \times \mathbb{R}^d$

- Rough fluctuations in time

- Homogeneous with respect to the space coordinate

- Defined on $(\Omega, \mathcal{F}, \mathbf{P})$

- Centered Gaussian process

- Covariance:

$$\mathbf{E}\left[B(t,x)B(s,y)\right] = (s \wedge t)\, Q(x-y)$$

- Q covariance function such that $Q(0) < \infty$

(iii) The polymer measure

- Hamiltonian at $t > 0$:

$$-H_t(\omega) = \int_0^t B(ds, \omega_s)$$

- For a fixed ω, $H_t(\omega) \sim \mathcal{N}(0, tQ(0))$.

- Gibbs polymer measure: $x \in \mathbb{R}^d$, $\beta > 0$

$$\begin{aligned} dG_t^x(\omega) &= \frac{e^{-\beta H_t(\omega)}}{Z_t^x} d\widehat{P}^x(\omega) \\ Z_t^x &= E_\omega^x \left[e^{-\beta H_t(\omega)} \right] \end{aligned}$$

Gibbs averages

- $t \geq 0$, $n \geq 1$
- $f : (C([0,t];\mathbb{R}^d))^n \to \mathbb{R}$ bounded

$$\langle f \rangle_t = \frac{E_\omega^x \left[f(\omega^1, \ldots, \omega^n) e^{-\beta \sum_{l \leq n} H_t(\omega^l)} \right]}{Z_t^n}$$

- $\omega^l, 1 \leq l \leq n$, independent Brownian configurations
- $\langle f \rangle_t$ is still a random variable in $(\Omega, \mathcal{F}, \mathbf{P})$

2) Questions

- Influence of the media B on the asymptotic behavior of ω

- Existence of a phase transition

- Limit theorems for Z_t

- Competent normalization for $\langle \omega_t \rangle_t$

- Diffusive or superdiffusive behavior of ω under G_t

3) Related models

1. Random walk in a discrete iid potential
   - Imbrie-Spencer, Bolthausen, Carmona-Hu
2. Brownian motion in a discretized potential
   - Conlon-Olsen, Coyle
3. Brownian motion in a Poisson potential
   - Comets-Yoshida
4. Lyapounov exponent for SPDEs
   - Carmona-Molchanov-Viens, Cranston-Mountford-Shiga, Tindel-Viens

4) Existing results

- Definition of a weak and strong disorder regime in terms of $\lim \frac{1}{t} \log(Z_t^x)$

- Transition from weak to strong disorder as β varies

- Diffusive behavior of ω in the weak disorder regime

- In the strong disorder regime

  1. Case of a Gaussian random walk in a discrete potential

  2. $\sup_{s \le t} |\omega_s| \sim t^{\xi(d)}$ (rough definition)

  3. $\xi(d) \le \frac{3}{4}$, $\xi(1) \ge \frac{3}{5}$

5) The free energy

- Define the free energy of the system:

$$p_t(\beta) = \frac{1}{t}\mathbf{E}\left[\log\left(Z_t^x\right)\right],$$

Proposition 1 *For all* $\beta > 0$ *there exists a constant* $p(\beta) > 0$ *such that*

$$p(\beta) \equiv \lim_{t\to\infty} p_t(\beta) = \sup_{t\geq 0} p_t(\beta).$$

Sketch of the proof: Use

- Markov property of ω

- Independence of the increments of B

- Subadditivity argument

Proposition 2 *The function p satisfies:*

1. *The following upper bound holds true:*

$$p(\beta) \leq \frac{\beta^2}{2} Q(0). \tag{1}$$

2. $\mathbf{P}$*-almost surely, we have*

$$\lim_{t \to \infty} \frac{1}{t} \log Z_t = p(\beta). \tag{2}$$

Sketch of the proof:

- For (1), Jensen's inequality

- For (2), concentration inequalities using Malliavin calculus

6) Weak and strong disorder

First natural definition of weak disorder:

$$p(\beta) = \frac{\beta^2 Q(0)}{2}$$

i.e. $$\lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\log(Z_t)\right] = \lim_{t\to\infty} \frac{1}{t}\log\left(\mathbf{E}\left[Z_t\right]\right)$$

Another definition:

- Set $W_t = Z_t \exp(-\frac{\beta^2 Q(0) t}{2})$
- W positive $\mathcal{F}_t$-martingale
- Set $W_\infty = \lim_{t\to\infty} W_t$
- $\mathbf{P}\left(W_\infty = 0\right) \in \{0, 1\}$
- $W_\infty > 0$ implies $p(\beta) = \frac{\beta^2 Q(0)}{2}$

Definition 3 *We will say that the polymer is in a strong disorder regime if* $W_\infty = 0$ *almost surely, while the weak disorder phase will be defined by* $W_\infty > 0$ *almost surely.*

7) Example of weak disorder

Assumption

(H) Q is a symmetric function from $\mathbb{R}^d$ to $\mathbb{R}$ and β a positive constant satisfying

$$E_\omega\left[\exp\left(\frac{\beta^2}{2} I_\infty(Q)\right)\right] < \infty,$$

$$\text{where} \qquad I_\infty(Q) = \int_0^\infty Q(\omega_s)ds$$

Proposition 4 *Under hypothesis (H), we have*

$$\mathbf{P}(W_\infty > 0) = 1 \qquad \textit{and} \qquad p(\beta) = \frac{\beta^2 Q(0)}{2}.$$

Sketch of the proof:

- L^2 computations for martingales

Proposition 5 *Assume*

- $d \geq 3$
- $Q(x) = \tilde{Q}(|x|)$, $\tilde{Q}$ *is a positive function from* $\mathbb{R}$ *to* $\mathbb{R}$
- β *is small enough*
- $\int_0^\infty x\tilde{Q}(x)dx < \infty$

Then hypothesis **(H)** *is satisfied.*

Sketch of the proof:

- Identities in law for Bessel(d)
- Fernique's lemma

8) Example of strong disorder

Example 6 *Assume*

- $d \geq 1$
- $c_1(1 \wedge |x|^{-\rho}) \leq Q(x) \leq c_2(1 \vee |x|^{-\rho})$
- $\rho < 1$

Then the polymer will be in the strong disorder regime for any value of $\beta > 0$.

Sketch of the proof:

- $\theta \in (0, 1)$
- Prove that $\lim_{t \to 0} \mathbf{E}[W_t^\theta] = 0$
- Itô's formula

9) Particle systems

- Obvious relation between G_t and Feynman-Kac models

- The approximation of G_t by a particle system seems easy in the discrete case

- The method designed by Del Moral and Miclo for the computation of Lyapounov exponents seems harder to adapt to the random continuous case

- Can we shed a light on these polymer measures using simulations?