Volatility of daily stock returns estimation by means of particle filter: The IBEX 35 case

M. P. Muñoz[1], D. Marquez[2], M. Martí-Recober[1], C. Villazón[2], L. Acosta[1]

[1] Statistics and Operational Research Dept. (UPC)
[2] Business Economic Dept. (UAB)

Person contact: pilar.munyoz@upc.es

UPC

UAB Universitat Autònoma de Barcelona

Index

1. Introduction

Volatility is an important characteristic of financial markets

Financial time series present:

- **Excess kurtosis**
- **Asymmetric effects of positive or negative shocks**
- **Small first-order autocorrelation of squared observations**
- **Slow decay towards zero of the autocorrelation coefficients of squared observations**

Variance responds asymmetrically to past returns (*Harvey and Shepard, 1996***): variance tends to be higher under influence of bad news than under influence of good news**

1. Introduction

Our approach in this work is :

- **To estimate the Spanish IBEX volatility by means of a model to capture simultaneously the mean and variance asymmetries in time series**

- **Particle filter is adopted for parameter estimation**

2. Models (i)

<u>Stochastic Volatility model</u> (*Taylor 1982*):

- flexible in capturing the excess kurtosis observed
- formulated with leverage effects, captures the asymmetric behaviour in stock returns

<u>GARCH</u> model (*Bollerslev 1986*):

- It describes volatility clustering and excess kurtosis (although not entirely)

<u>TAR</u>: Threshold Autoregressive model (*Tong, 1990*):

- It is a piecewise AR model
- The switching mechanism is controlled by the delayed process variable (threshold variable)
- It captures the asymmetric patterns of volatility

2. Models (ii)

Previous models can be combined to produce "second-generation" models:

- **<u>Threshold GARCH</u> (TGARCH)** *(Rabemananjara and Zakoïan, 1993):*
  - Threshold non-linearity is incorporated into the GARCH variance specifications
- **<u>Threshold Stochastic Volatility Model</u>** *(So, Li and Lam, 2002)*
  - Threshold non-linearity is incorporated into the Stochastic Volatility:
    - in the mean
    - in the variance
- **And so on …**

2.1 Basic Stochastic Volatility model (i)

The known model is:

$$y_t = \sigma_t \varepsilon_t \qquad (1a)$$

- y_t : the observed variable, in general the return on an asset
- σ_t : the unobserved volatility of y_t

Volatility evolution supposed to be governed by:

$$\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \sigma_\nu \nu_t \qquad (1b)$$

- ε_t **and** ν_t **:** Gaussian white sequences, independent, with mean 0 and variance 1
- β: model persistence

2.1 Basic Stochastic Volatility model (ii)

In state space representation:

- **The observation equation**

$$y_t = \exp(0.5x_t)\varepsilon_t \qquad (2a)$$

- **The system equation:**

$$x_t = \alpha + \beta x_{t-1} + \sigma_\nu \nu_t \qquad (2b)$$

  - **$x_t = \log \sigma_t^2$, a latent variable**

- It describes a discrete time, non-linear and Gaussian dynamic system
- The non-linearity appears only in the observation equation
- It evolves as a first-order Markov process

2.1 Basic Stochastic Volatility model (iii)

- **Main goal: To estimate the system state :**

  $x_t = \log \sigma_t^2$ $\longleftrightarrow$ Volatility

- **Implicitly: To** estimate the unknown parameters

$$\theta = (\alpha, \beta, \sigma_v)$$

- Different estimation methods have been developed (*Broto and Ruiz, 2002*):
  - Method of moments
  - Quasi-maximum likelihood
  - Bayesian methods

2.2 Threshold Stochastic Volatility model (i)

THSV model (*So, Li and Lam, 2002*):

- Define a set of Bernoulli random variables $\mathbf{s_t}$ by:

$$s_t = \begin{cases} 0 & if\ r_{t-1} < 0 \\ 1 & if\ r_{t-1} \geq 0 \end{cases}$$

r_t: returns

- Model:

$$r_t = \phi_{0s_t} + \phi_{1s_t} r_{t-1} + y_t$$

$$y_t = \sigma_t \varepsilon_t \quad \varepsilon_t \sim N(0,1)$$

$$\log \sigma_t^2 = \alpha_{s_{t+1}} + \beta_{s_{t+1}} \log \sigma_{t-1}^2 + \sigma_v v_t \quad v_t \sim N(0,1)$$

ε_t and ν_t : Stochastically independent

The unknown parameters $\theta = (\phi_o, \phi_1, \alpha, \beta, \sigma_v)$

switch between the two regimes corresponding to the rise and fall in the asset prices.

This model can also be formulated into state space form

2.3 Bayesian State space approach (i)

- **Bayesian approach: To include the parameters as part of the state vector**

$$\begin{bmatrix} x_t \\ \theta \end{bmatrix}$$

- **From a bayesian point of view:** To obtain the a posteriori PDF

$$p(x_t, \theta \mid D_t)$$

Where $D_t = \{y_1, \ldots, y_t\}$ represents the available information until time t

2.3 Bayesian State space approach (ii)

It is well known that using recursive formula based on Bayes' rule is used

Assuming that the a priori PDF at time *t-1* is knowing

$$p(x_{t-1}, \theta \mid Y_{t-1})$$

- **State PDF prediction:**

$$p(x_t, \theta \mid Y_{t-1}) = \int p(x_t \mid x_{t-1}, \theta)\, p(x_{t-1}, \theta \mid Y_{t-1})\, dx_{t-1} \quad (3)$$

$p(x_t \mid x_{t-1}, \theta)$: State evolution density obtained from (2b)

- **Filtering PDF**

$$p(x_t, \theta \mid Y_t) \propto p(y_t \mid x_t, \theta)\, p(x_t, \theta \mid Y_{t-1}) \quad (4)$$

$p(y_t \mid x_t, \theta)$ Likelihood obtained from (2a)

2.4 Particle filtering

- **Filtering PDF is approximated by an empirical distribution formed from *particles* (point masses) in two steps (*Liu and West, 2001*):**

Prediction PDF approximation:

$$p(x_t,\theta \mid Y_{t-1}) \sim \sum_{j=1}^{M} p(x_t \mid x_{t-1}^{(j)},\theta)\tilde{w}_{t-1}^{(j)} \qquad (5)$$

Filtering PDF approximation:

$$p(x_t,\theta \mid Y_{t-1}) \propto p(y_t \mid x_t,\theta)\sum_{j=1}^{M} p(x_t \mid x_{t-1}^{(j)},\theta)\tilde{w}_{t-1}^{(j)} \qquad (6)$$

In this case:

$$\tilde{w}_t^{(j)} \propto p(y_t \mid x_t^{(j)},\theta) \qquad (7)$$

These recursions can be implemented using **Sampling Importance Resampling Algorithm (SIR)** (*Arulampalam et al., 2002)*

2.5 SIRJ Algorithm (i)

Using SIR, after few iterations, there is an impoverishment problem for the parameters.

- **Our approach SIRJ:** Modifies the Sampling Importance Resampling (SIR) procedure adding at the end of each iteration, the jitter proposed by Liu and West (2001). (*Muñoz et al. 2004*)
- We have compare SIRJ with the approach of Liu and West (2001). The SIRJ gives almost the same precision as de second one for the parameters estimation and it is computationally less expensive
- The algorithm is implemented using the R language for Statistical Computing (http://cran.r-project.org/)

2.5 SIRJ Algorithm (ii)

Pseudo-code Sequential Importance Resampling with Jiiter (SIRJ) Algorithm

$\left[\{x_t^{(j)}, w_t^{(j)}, \theta_t^{(j)}\}_{j=1}^{M}\right] = SIRJ\left[\{x_{t-1}^{(j)}, w_{t-1}^{(j)}, \theta_{t-1}^{(j)}\}_{j=1}^{M}, y_t\right]$

1. Initialization, t=0
   - FOR j=1:M
     Sample $x_0^{(j)} \sim p(x_0)$
     Sample $\theta_0^{(j)} \sim p(\theta_0)$
   - END FOR

- FOR t=1:N

2. Importance sampling
   - FOR j=1:M
     Prediction: Sample $x_t^{(j)} \sim q(x_t^{(j)} | x_{t-1}^{(j)}, \theta_t^{(j)}, y_t) = p(x_t^{(j)} | x_{t-1}^{(j)}, \theta_t^{(j)})$
     Filtering: Assign to each particle $x_t^{(j)}$ the new weight $w_t^{(j)}$ according to (7)
     - This means that $w_t^{(j)} \propto N(y_t \mid 0, \exp(x_t^{(j)}))$
   - END FOR
   - FOR j=1:M
     Normalize the importance weights:
     $$\tilde{w}_t^{(j)} = \frac{w_t^{(j)}}{\sum_{i=1}^{M} w_t^{(i)}}$$
   - END FOR

2.5 SIRJ Algorithm (iii)

3. Resampling
   - Resampling with replacement the particles $\{x_t^{(j)}, \theta_{t-1}^{(j)}, j = 1, \ldots, M\}$ with the importance weights $\{\tilde{w}_t^{(1)}, \ldots, \tilde{w}_t^{(M)}\}$

4. Jitter
   - For j=1:M
     Sample a new parameter vector
     $\theta_t^{(j)} \sim N(\cdot \mid m_t^{(j)}, h^2 V_t)$
     $m_t^{(j)} = a\theta_t^{(j)} + (1-a)\bar{\theta}_t$
     $\bar{\theta}_t, V_t$ mean and variance of the $p(\theta \mid D_t)$ Monte Carlo approximation
     a around 0.95 – 0.995
     $h^2 = 1 - a^2$
   - END FOR

END FOR

3. The IBEX 35 case (i)

IBEX 35 is the Spanish stock-exchange index

The prices series is observed daily from 02/01/1990 to 17/03/04 (3539 observations) (Fig. 1):

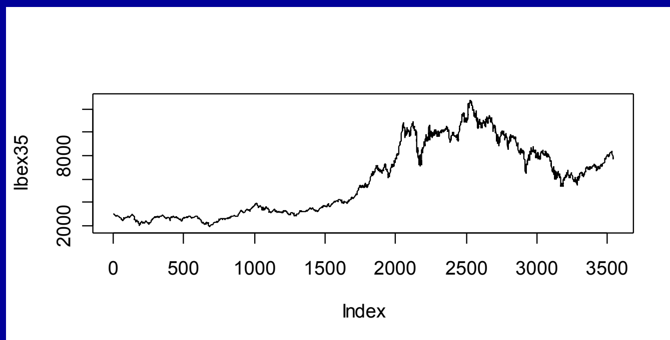


Fig.1: IBEX 35 Stock Index prices observed daily (02/01/1990 to 17/03/2004

3. The IBEX 35 case (ii)

The prices ($\boldsymbol{p_t}$) are transformed to returns ($\boldsymbol{r_t}$) by $r_t = 100 \cdot \log(p_t / p_{t-1})$

• Main features of returns series (i):

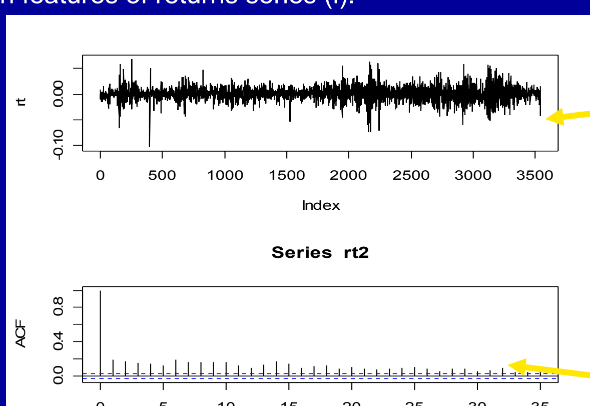


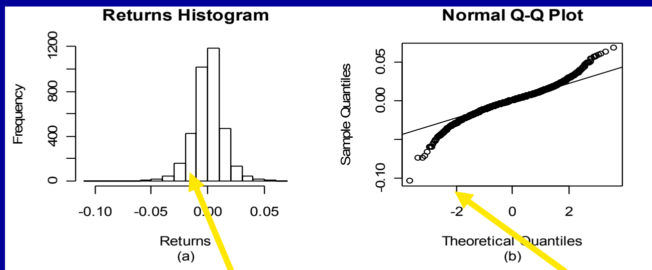
Fig.2: IBEX 35 Stock Index returns observed daily (02/01/1990 to 17/03/2004 and correlograma of squared returns

It exhibits volatility clusters

The squared observations are correlated

3. The IBEX 35 case (iv)

- Main features of returns series (ii):



Skewness= -0.237

Kurtosis = 6.339

Non normality

3. The IBEX 35 case (v)

Models fitted:

3.1. Basic stochastic volatility (i)

$$y_t = \sigma_t \varepsilon_t$$

$$\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \sigma_\nu \nu_t$$

Estimated parameters by means of the SIRJ procedure :

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}_w$
0.010 (0.003)	0.970 (0.004)	0.192 (0.020)

( ) St. dev

3. The IBEX 35 case (vi)

3.1. Basic stochastic volatility (ii)

<u>Model Checking</u>

The model diagnostics were based on the standardized observations, defined as $\hat{\varepsilon}_t = \hat{y}_t / \hat{\sigma}_t$ called "*residuals*"

$\hat{\sigma}_t$ is the volatility estimated, obtained by substituting the estimated parameters in the model equation (1b)

Descriptive statistics of standardized observations:

$\hat{\varepsilon}_t = \hat{y}_t / \hat{\sigma}_t$	Mean	S. Dev.	Skew.	Kurto	Q(20)	$Q_2(20)$	n
	0.028	0.976	-0.179	2.852	55.870*	27.322	3539

Q(20) and Q2(20): Box-Ljung statistics for observations and squared observations

* Significant at the 95% level

Autocorrelation between the observations

3. The IBEX 35 case (vii)

3.1. Basic stochastic volatility (iii)

<u>Conclusion</u>

Stochastic volatility model captures the Kurtosis of the IBEX returns but:

- Does not capture completely the Skewness
- There is autocorrelation between observations

3. The IBEX 35 case (viii)

3.2. Threshold stochastic volatility model (i)

Our approach (i):

A) Identify and estimate a threshold model (SETAR) for the mean following the methodology proposed by Tsay (1989) and use the algorithm designed by Márquez (2002)

- The model obtained is the following:

$$r_t = \begin{cases} \kappa_1 + \phi_1^1 r_{t-1} + \cdots + \phi_{15}^1 r_{t-15} + y_t, & r_{t-1} < 0 \\ \kappa_2 + \phi_1^2 r_{t-1} + \cdots + \phi_{15}^2 r_{t-15} + y_t, & r_{t-1} \geq 0 \end{cases}$$

SETAR (2;15,13)	$\hat{\phi}_3$	$\hat{\phi}_4$	$\hat{\phi}_6$	$\hat{\phi}_8$	$\hat{\phi}_{11}$	$\hat{\phi}_{13}$	$\hat{\phi}_{15}$	s_R	AIC	n
1r. reg.	-0.0507 (0.0259)	0	0	0.0602 (0.0258)	0	0	0.0977 (0.0263)	2.13	1288.96	1662
2º. Reg.	0	0.0478 (0.0214)	-0.0436 (0.0219)	0.0454 (0.0220)	0.0507 (0.0217)	0.0529 (0.0217)	0	1.61	920.74	1862

3. The IBEX 35 case (ix)

3.2. Threshold stochastic volatility model (ii)

Our approach (ii):

B) Fit to the residuals obtained from the SETAR model, a threshold stochastic volatility model for the variance. The parameter estimation will be made by means the SIRJ procedure

$$s_t = \begin{cases} 0 & if\ r_{t-1} < 0 \\ 1 & if\ r_{t-1} \geq 0 \end{cases}$$

$$y_t = \sigma_t \varepsilon_t \quad \varepsilon_t \sim N(0,1)$$

Now, y_t are the residuals from the SETAR model

$$\log \sigma_t^2 = \alpha_{s_{t+1}} + \beta_{s_{t+1}} \log \sigma_{t-1}^2 + \sigma_v v_t \quad v_t \sim N(0,1)$$

Estimated parameters

$\hat{\beta}$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\sigma}_w$
0.962 (0.002)	0.079 (0.009)	-0.037 (0.009)	0.172 (0.014)

3. The IBEX 35 case (x)

3.2. Threshold stochastic volatility model (iii)

Model checking

The residual analysis for this model is satisfactory :

$\hat{\varepsilon}_t = \hat{y}_t / \hat{\sigma}_t$	Mean	S. Dev.	Skew.	Kurtosi	Q(20)	$Q_2(20)$	n
	0.007	0.955	-0.112	2.634	15.705	25.459	3509

- The model captures the asymmetric behaviour in the IBEX returns and the excess-kurtosis observed
- The residuals and the squared residuals are not autocorrelated
- The residuals follow a Gaussian distribution

3. The IBEX 35 case (xi)

3.3. Summary

First generation models	Returns equation	Volatility equation
SV	$r_t = \sigma_t u_t \quad u_t \sim NID(0,1)$	$\ln(\sigma_t^2) = 0.01 + 0.97\ln(\sigma_{t-1}^2) + 0.192 w_t \quad w_t \sim NID(0,1)$
SETAR	$r_t = \begin{cases} -0.05 r_{t-3} + 0.06 r_{t-8} + 0.10 r_{t-15} + y_t, & r_{t-1} < 0 \\ 0.05 r_{t-4} - 0.04 r_{t-6} + 0.05 r_{t-8} + 0.05 r_{t-11} + 0.05 r_{t-13} + y_t, & r_{t-1} \geq 0 \end{cases}$	-
Second generation model		
SETAR-THSV	$r_t = \begin{cases} -0.05 r_{t-3} + 0.06 r_{t-8} + 0.10 r_{t-15} + y_t, & r_{t-1} < 0 \\ 0.05 r_{t-4} - 0.04 r_{t-6} + 0.05 r_{t-8} + 0.05 r_{t-11} + 0.05 r_{t-13} + y_t, & r_{t-1} \geq 0 \end{cases}$	$\ln(\sigma_t^2) = \begin{cases} 0.079 + 0.962\ln(\sigma_{t-1}^2) + 0.172 w_t & r_{t-1} < 0 \\ -0.037 + 0.962\ln(\sigma_{t-1}^2) + 0.172 w_t & r_{t-1} \geq 0 \end{cases}$ $w_t \sim NID(0,1)$

4. Conclusions

- **SIRJ works well with both models and it is easy to implement.**
- **SETAR-THSV model:**
    - o **can be implemented using bayesian methods and gives good results.**
    - o **captures the dynamic behaviour of the returns and volatility series.**
    - o **captures the asymmetries and reduces the excess of kurtosis in the volatility series**

References (i)

- Arulampalan MS, Maskell S, Gordon N, Clapp TA. 2002. Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking. IEEE Transactions on Signal Processing **50**: 174-188
- Bollerslev T. 1986. Generalized autoregressive conditional heterocedasticy. *Journal of Econometrics* **31**: 307-327.
- Broto C, Ruiz E. 2002. Estimation methods for Stochastic Volatility. *Working Paper 02-54 (14). Universidad Carlos III de Madrid*.
- Harvey AC, Shepard N. 1996. Estimation of an asymmetric stochastic volatility model for asset returns. *Journal of Business and Economic Statistics* **14**: 420-434
- Liu J, West M. 2001. Combined parameters and state estimation in simulation-based filtering. In Sequential Monte Carlo Methods in Practice. Doucet A, de Freitas M, Gordon N (eds). Springer-Verlag.
- Márquez MD. 2002. Modelo SETAR aplicado a la volatilidad de la rentabilidad de las acciones: algorítmos para su identificación. *Ph D dissertation. Universitat Politècnica de Catalunya*.

References (ii)

- Muñoz MP, Márquez MD, Martí-Recober M, Villazón C, Acosta L. 2004. TAR-GARCH and Stochastic Volatility Models: Evaluation Based on Simulations and Financial Time Series. COMPSTAT 2004. (Accepted as Contributed Paper)
- Rabemananjara M, Zakoïan RY. 1993. Threshold ARCH and asymmetries in volatility. *Journal of Applied Econometrics* **8**
- So MK, Li WK, Lam K. 2002. A Threshold Stochastic Volatility Model. *Journal of Forecasting* **21**: 473-500.
- Taylor SJ. 1982. Financial returns modelled by the product of two stochastic process, a study of daily sugar prices 1961-79. In *Time Series Analysis: Theory and Practice 1*. Anderson OD (ed). North-Holland: 203-226.
- Tong H. 1990. *Non linear time series: A dynamical system approach*. Oxford University Press.
- Tsay R. 1989. Testing and modelling Threshold Autoregressive Processes. Journal of the American Statistical Association **84**: 231-240.