# Reserve Price in Progressive Second Price Auctions*

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**Abstract.** Pricing has become mandatory to overcome congestion and to offer service differentiation in communication networks. Whereas many pricing schemes have been designed in the literature, we focus here on the so-called Progressive Second Price Auction that allocates bandwidth on an auction-basis: users sequentially declare the amount of bandwidth they expect and how much they value it. At each time, the network allocates bandwidth to users with the highest willingness-to-pay and charge them with the bid of those excluded from the game by their presence. Convergence, efficiency and incentive compatibility have been verified in the literature for this scheme. Nevertheless one degree of freedom still remains in the model, namely the reserve price that is the minimal unit price at which the network accepts to sell the bandwidth. We propose here to determine the reserve price maximizing the network revenue. This analysis is based on the assumption that the demand functions and distribution of the (*a priori* random) number of users in the network are known.

## 1 Introduction

Pricing for bandwidth has become an important topic in telecommunication networks, mainly for two reasons. First, demand for bandwidth keeps increasing, creating congestion that degrades the quality of service [3]. Second, telecommunication networks have to supply different kinds of services, with different quality of service (QoS) requirements. Pricing would permit to limit the congestion problem and to provide incentives so that all users will not choose

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the best QoS. For surveys on pricing schemes in telecommunication networks, the reader can see for instance [1, 4, 17] and the references therein.

One of the main streams of pricing schemes is auctioning. In [11], McKie-Mason and Varian suggest the use of a *smart market* where a bid is associated to each packet and those with the highest bid are given the highest priority. The price is computed based on the Vickrey-auction principle, as the bid of the lowest priority packet admitted. In [6, 14], the costly per packet auctions are replaced by auctions for bandwidth during intervals of time, called Progressive Second Price Auctions (PSP). Elastic users sequentially submit bids composed of the amount of bandwidth they require and the unit price they propose. Their bid depends on the bids already in competition for the resource. The bids with the highest unit price are allocated the bandwidth they ask and the total charge imposed to a user is computed as the accumulated bid prices of those excluded by his presence. To make sure that the resource is not sold at a too low level, a *reserve price* has been introduced, corresponding to a minimum unit price. It is then shown that, under some assumptions, the game converges to an equilibrium that is efficient in terms of social welfare. This work has been extended in [15] to the analysis of a whole network, and similar convergence and optimality properties have been shown. Other extensions and further analysis of the PSP can be found in [2, 5, 8–10, 16].

Whereas PSP analysis has been extensive, there remains the question about how to fix the reserve price, the last degree of freedom of the model. We investigate this issue in this paper, in order to maximize the expected network revenue. Based on the result of [7] that shows that, in steady-state and if demand exceeds capacity, the network revenue is close to the unit price multiplied by the total capacity of the resource, we propose to determine the optimal reserve price when the number of players is *a priori* unknown, but its distribution is known.

The layout of the paper is as follows. Section 2 recalls the basic results on the PSP. Section 3 looks at the reserve price in terms of the market clearing price and presents how it can be chosen so that the revenue is maximized. Section 4 illustrates the results that can be obtained and conclusions are presented Section 5.

## 2 Progressive Second Price Mechanism

We summarize here the main results of [6].

Consider a single resource, with capacity $Q$. Assume that $I$ players compete for it in an auction process, where the players bid sequentially. Player $i$'s bid is $s_i = (q_i, p_i)$ where $q_i$ is the capacity player $i$ is asking and $p_i$ is the unit price he is proposing. Let $s = (s_1, \cdots, s_I)$ be the bid profile and $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_I)$ be the profile where player $i$'s bid is excluded. We will sometimes write $s = (s_i; s_{-i})$ in order to emphasize player $i$'s bid. For $y \geq 0$ define

$$\underline{Q}_i(y; s_{-i}) = \left[ Q - \sum_{k \neq i\ : p_k \geq y} q_k \right]^+ .$$

The progressive second price allocation rule [6, 14] gives to player $i$ a bandwidth

$$a_i(s) = \min(q_i, \underline{Q}_i(p_i; s_{-i})), \quad (1)$$

so that the highest bids are allocated the desired quantity, and the total cost is given by the declared willingness to pay (bids) of the users who are excluded by $i$'s presence, i.e.,

$$c_i(s) = \sum_{j \neq i} p_j [a_j(0; s_{-i}) - a_j(s_i; s_{-i})]. \quad (2)$$

It is assumed that a fee $\varepsilon$ is charged each time a player submits a bid and that player $i$ has a budget constraint $b_i$ which imposes her that $c_i(s_i, s_{-i}) \leq b_i$. Let $\mathcal{S}_i(s_{-i})$ be the set of player $i$'s bids verifying this constraint.

Assume that player $i$ attempts to maximize a *quasi-linear utility function* $u_i(s) = \theta_i(a_i(s)) - c_i(s)$, where $\theta_i$ is player $i$'s valuation function, i.e. $\theta_i(a_i(s))$ is the price that player $i$ is willing to pay to obtain the allocation $a_i(s)$.

Also, a bid $s_0 = (Q, p_0)$ is introduced, meaning that the seller will allocate bandwidth at a minimum unit price $p_0$, which is called the *reserve price*. The seller can thus be seen as a player (not in $\mathcal{I}$) with a valuation function $\theta_i(q) = p_0 q$.

Under some smoothness assumptions over functions $\theta_i$, the following properties are shown:

- **Incentive Compatibility**. We say that a bid $s_i = (q_i, p_i)$ is *truthful* if it verifies $p_i = \theta_i'(q_i)$.
  Let

  $$Q_i(y; s_{-i}) = \left[ Q - \sum_{k \neq i\ : p_k > y} q_k \right]^+$$

  and

  $$G_i(s_{-i}) = \sup \left\{ z : z \leq Q_i(\theta_i'(z), s_{-i}) \text{ and } c_i(z) \leq b_i \right\}.$$

  $\forall 1 \leq i \leq I, \forall s_{-i}$ such that $Q_i(0, s_{-i}) = 0$, $\forall \varepsilon > 0$, there exists a truthful $\epsilon$-best reply, that is a truthful bid $t_i = (v_i, w_i)$ that ensures $i$ to get within $\epsilon$ of the best possible utility:

  $$t_i(s_{-i}) = \left(v_i = \left[G_i(s_{-i}) - \varepsilon/\theta_i'(0)\right]^+, \omega_i = \theta_i'(v_i)\right)$$

– **Convergence**. If all the players bid like described above, the game converges to a $2\epsilon$-Nash equilibrium, where an $\epsilon$-Nash equilibrium is a bid profile $s$ such that $\forall i \in \mathcal{I}$,

$$\begin{cases} s_i \in \mathcal{S}_i(s_{-i}) \\ u_i(s_i\ ; s_{-i}) \geq u_i(s'_i\ ; s_{-i}) - \epsilon, \forall s'_i \in \mathcal{S}_i(s_{-i}) \end{cases}$$

– **Optimality**. For the previous $2\epsilon$-Nash equilibrium, the resulting overall utility $\sum_{i\in\mathcal{I}\cup\{0\}} \theta_i(a_i)$ is maximized.

# 3 Reserve price and optimization problem

## 3.1 Reserve price and market clearing price

In [7], we studied the equilibria that can be reached by the PSP mechanism when demand exceeds supply, that is when

$$\sum_{i\in\mathcal{I}} d_i(p_0) > Q, \tag{3}$$

where $p_0$ is the reserve price fixed by the resource seller, and $d_i$ is player $i$'s *demand function*, i.e.

$$d_i(p) = \arg\max_q\{\theta_i(q) - pq\}.$$

We proved in [7] that under condition (3), the seller can ensure a revenue close to $p_0 Q$ as the bid fee $\epsilon$ tends to 0[4].
If (3) is not verified, then all users $i \in \mathcal{I}$ will ask for the quantity $d_i(p_0)$ of resource they wish to get when the unit selling price is $p_0$. As a result, the total quantity of resource that will be sold is $d(p_0) = \sum_{i\in\mathcal{I}} d_i(p_0)$, bringing the seller the revenue $p_0 d(p_0)$.

## 3.2 Reserve price and revenue

In consequence, the Progressive Second Price mechanism applied with a small bid fee $\varepsilon$ ensures the seller to receive a revenue close to

$$R(p_0, \mathcal{I}) = p_0 \times \min\left(Q, d(p_0)\right) \tag{4}$$

*Remark 1.* We are faced here with the same trade-off as for a single-item auction (see [12]) : increasing the reserve price may increase the selling price, but may also reduce the probability that all the resource be sold.

[4] We proved in [7] that this value is the minimum revenue that the network can expect. However, depending on the valuation functions of users and on the order of bids among them, the revenue may be higher (see [9]). Since the network cannot control these parameters, we assume that she will try to maximize this minimum value

*Remark 2.* If the seller chooses to sell the resource at a fixed unit price $p_0$ without using the PSP mechanism, then his revenue will also be $p_0 \times \min(Q, d(p_0))$. However, when demand exceeds supply, each user will want to buy a quantity $d_i(p_0)$ of resource until all the capacity is allocated, and therefore the allocation will not be efficient in the sense of social welfare.

The value $R(p_0, \mathcal{I})$ depends on the set of players $\mathcal{I}$ through the relation $d(p_0) = \sum_{i \in \mathcal{I}} d_i(p_0)$. But this set is *a priori* unknown. Assume that there are $T$ different types of users (which might correspond to $T$ different types of applications). Assume that we know from observations the joint distribution $\mathbb{P}_n$ of the number of players competing for bandwidth, with $\mathbb{P}_n((n_1, ..., n_T))$ being the probability that there are $n_t$ players of type $t$ $(1 \leq t \leq T)$. Let $d_{(t)}(p)$ correspond to the demand function of a type-$t$ user and $\theta_{(t)}$ be her valuation function. The expected network revenue can then be expressed using the joint law $\mathbb{P}_n$ by

$$\mathbb{E}R(p_0) = p_0 \times \sum_{(n_1,...,n_T) \in \mathbb{N}^T} \mathbb{P}_n((n_1, ..., n_T)) \min\left(Q, \sum_{t=1}^{T} n_t d_{(t)}(p_0)\right). \tag{5}$$

We now assume that the seller will fix her reserve price $p_0$ so as to maximize this expected revenue.

### 3.3 Optimization of the expected revenue

In this section, we study the concavity properties of the expected revenue $\mathbb{E}R(.)$ under a concavity assumption on the demand functions $d_{(t)}, 1 \leq t \leq T$. These properties imply that the determination of the reserve price that maximizes the expected revenue can easily be obtained.

**Assumption A** *For all $t, 1 \leq t \leq T$, the demand function $d_{(t)}$ is such that $p \to p d_{(t)}(p)$ is concave on $[0, \theta'_{(t)}(0)]$.*

For example, quadratic valuation functions of the form

$$\theta_{(t)}(z) = -\kappa_t(\min(z, \bar{q}_t))^2/2 + \kappa_t \bar{q}_t \min(z, \bar{q}_t),$$

where $\bar{q}_t$ is the line rate, lead to demand functions $d_{(t)}(p) = [\theta'_{(t)}(0) - p]^+/a_t$ (with $\theta'_{(t)}(0) = \kappa_t \bar{q}_t$) verifying Assumption A.

**Proposition 1.** *We assume without loss of generality that $\theta'_{(1)}(0) \geq \theta'_{(2)}(0) \geq ... \geq \theta'_{(T)}(0)$. Under assumption A, the expected revenue $\mathbb{E}R(p_0)$ is a concave function of the reserve price $p_0$ on every segment of the form $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)], 1 \leq t \leq T$ (where $\theta'_{(T+1)}(0) = 0$).*

*Proof.* Consider a fixed $t$. On interval $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)], 1 \le t \le T$, all demand functions $d_{(r)}$ are such that function $p \to pd_{(r)}(p)$ is concave. Indeed, if $r > t$ then $d_{(r)} = 0$ if $p_0 \in [\theta'_{(t+1)}(0), \theta'_{(t)}(0)]$, whereas if $r \le t$ then the concavity comes from Assumption A.
As a result, $\forall n = (n_1, ..., n_T) \in \mathbb{N}^T$, the function $p \to p \sum_{r=1}^{T} n_r d_{(r)}(p)$ is concave on $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)]$, as a finite sum of concave functions.
We now establish the concavity of the expected revenue on interval $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)]$: let $(p_1, p_2) \in [\theta'_{(t+1)}(0), \theta'_{(t)}(0)]^2$, and $\lambda \in [0, 1]$. We have

$$\begin{aligned}
\mathbb{E}R(\lambda p_1 + (1-\lambda)p_2) &= \mathbb{E}\ \min\Big( Q[\lambda p_1 + (1-\lambda)p_2], \\
&\qquad [\lambda p_1 + (1-\lambda)p_2] \sum_{r=1}^{T} n_r d_{(r)}(\lambda p_1 + (1-\lambda)p_2)\Big) \\
&\ge \mathbb{E}\ \min\Big( Q\lambda p_1 + Q(1-\lambda)p_2, \\
&\qquad \lambda p_1 \sum_{r=1}^{T} n_r d_{(r)}(p_1) + (1-\lambda)p_2 \sum_{r=1}^{T} n_r d_{(r)}(p_2)\Big) \\
&\ge \mathbb{E}\ \Big[\min\Big( Q\lambda p_1, \lambda p_1 \sum_{r=1}^{T} n_r d_{(r)}(p_1)\Big) \\
&\qquad + \min\Big( Q(1-\lambda)p_2, (1-\lambda)p_2 \sum_{r=1}^{T} n_r d_{(r)}(p_2)\Big)\Big] \\
&\ge \lambda \mathbb{E}R(p_1) + (1-\lambda)\mathbb{E}R(p_2)
\end{aligned}$$

where the second line comes from the concavity of the function $p \to p \sum_{r=1}^{T} n_r d_{(r)}(p)$ and the non-decreasingness of $x \to \min(K, x)$ for all fixed $K$, and the third line is a consequence of the inequality $\min(a + b, c + d) \ge \min(a, c) + \min(b, d)$ for all $(a, b, c, d) \in \mathbb{R}^4$.

The following corollary is then straightforward, recalling that the revenue is necessarily null on $[\theta'_{(1)}(0), +\infty]$ for all demand functions are null:

**Corollary 1.** *The reserve price $p_0$ maximizing the expected revenue can be determined by finding out the (unique) optimal value on each of the (finite number of) intervals $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)]$ and by comparing them.*

## 4 Numerical illustration

As an illustration of Proposition 1 and Corollary 1, consider the case where $T = 3$, and where the random variables representing the number of players $n_1, n_2, n_3$ of each type follow independent Poisson distributions, each with mean 4. The demand functions (and the corresponding marginal valuation functions) are supposed to be

$$\begin{aligned}
d_{(1)}(p) &= [5 - p/2]^+ \Leftrightarrow \theta'_{(1)}(q) = [10 - 2q]^+ \\
d_{(2)}(p) &= [7 - p]^+ \quad\ \Leftrightarrow \theta'_{(2)}(q) = [7 - q]^+ \\
d_{(3)}(p) &= [10 - 2p]^+ \Leftrightarrow \theta'_{(3)}(q) = [5 - q/2]^+.
\end{aligned}$$

Figure 1 displays the demand functions, expected demands (independently of $Q$), expected $\min(Q, d)$ and expected revenue in terms of the reserve price. The different intervals $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)]$

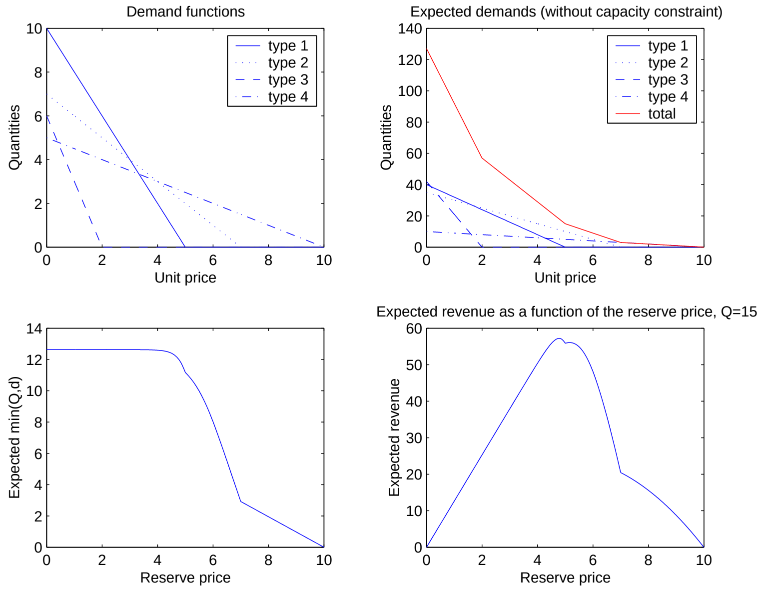


**Fig. 1.** Expected revenue as a function of the reserve price, for $T = 3$ and quadratic valuation functions

are $[0, 5]$, $[5, 7]$ and $[7, 10]$. ¿From Proposition 1, $p \to \mathbb{E}R(p)$ is concave on each $[\theta'_{(t+1)}(0), \theta'_{(t)}(0)]$, so that there is a unique maximum, which can easily be determined by standard numerical optimization tools [13]. Using Corollary 1, by comparing the value over each interval, the global optimum can be obtained. On our example, the reserve price providing the best revenue is obtained on $[5, 7]$ at $p_0 = 5.5$, giving an average revenue of 67.9.

## 5 Conclusions

Progressive Second Price Auctions are an efficient way to allocate bandwidth among users. In this auction model, there remained a last degree of freedom, the reserve price that is the smallest unit price at which the seller accepts to allocate bandwidth. Based on a model where the actual number of users is unknown (but its distribution is known), we have provided a method so that the reserve price maximizing the average seller's revenue is easily obtained.

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