Classes of Rational Graphs

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Irisa

Journées Montoises 2006

Plan

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Graphs

Graphs

A graphe G is a subset of $V \times A \times V$

- V is an arbitrary set of vertices
- A is a finite set of labels

Conventions

(u, a, v) is denoted by $u \xrightarrow{a} v$

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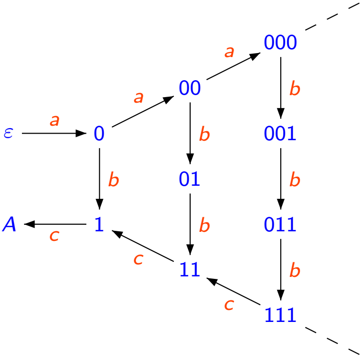
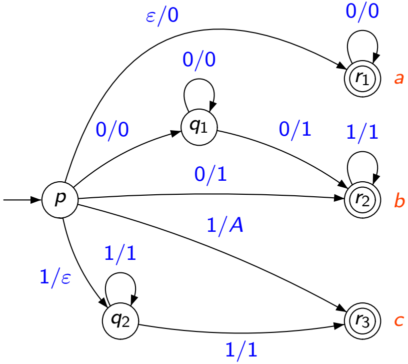
Rational graphs

Definition

A rational graph is defined by a rational transducer





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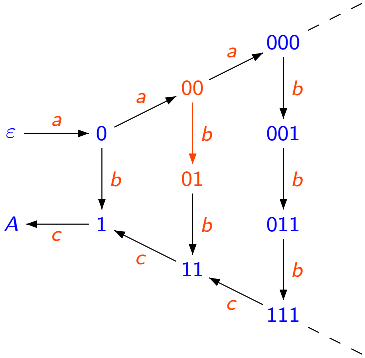

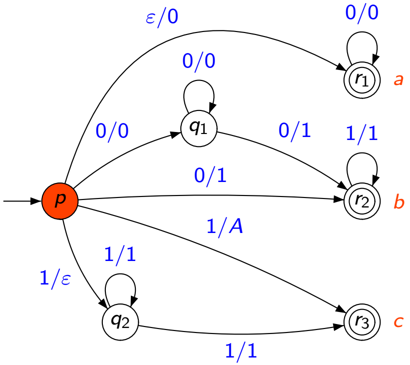

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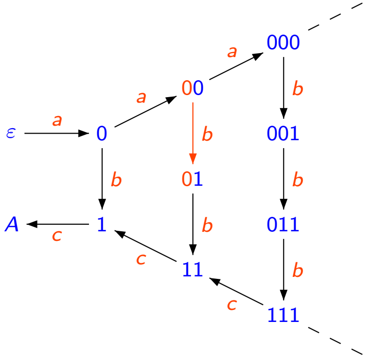

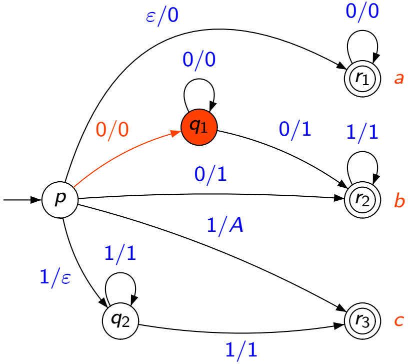

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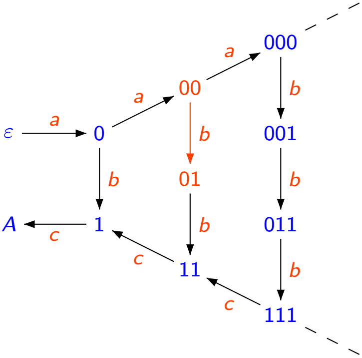

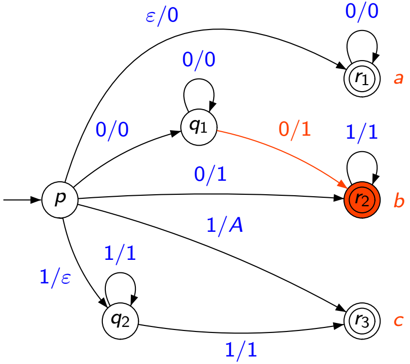

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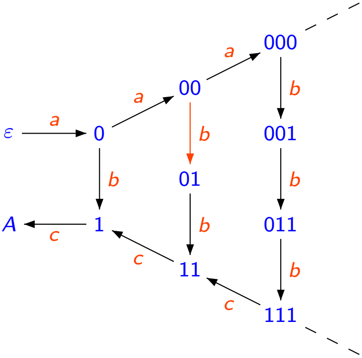
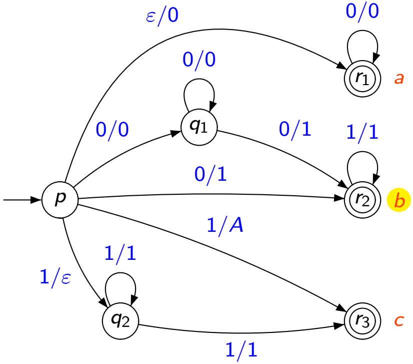




Classical questions

Proposition

Following properties are undecidable for rational graphs:

(i) *Accessibility (between two vertices);*

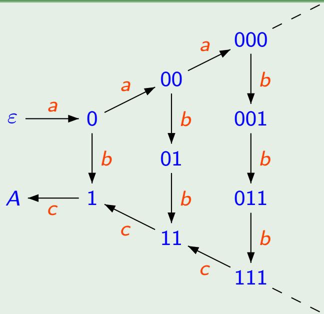
(ii) *Connectedness (of the whole graph);*

(iii) *Isomorphism (of two graphs);*

(iv) *First order theory (of a rational graph).*

Traces

Example (traces)



Theorem (Morvan Stirling 01)

The traces of rational graphs between two vertices coincide with the context sensitive languages.

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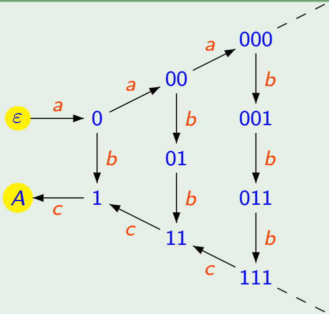

The trace of this graph between ε and A is $\{a^n b^n c^n \mid n \in \mathbb{N}\}$

$$Tr(G, \varepsilon, A) = \{a^n b^n c^n \mid n \in \mathbb{N}\}$$

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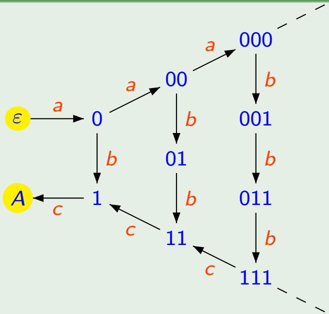

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Deterministic Rational Graphs

Definition

A rational graph is deterministic if each vertex is the source of at most on arc for each label

Proposition

Determinism is decidable for rational graphs

Proposition

First order theory and accessibility are undecidable for deterministic rational graphs

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The traces of deterministic rational graphs form a boolean algebra of deterministic context-sensitive languages containing non context-free languages.

Question

Are there context-free languages not contained in the traces of deterministic rational graphs?

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Rational Trees

Definition (Tree)

- at most one ancestor per vertex — decidable
- a single root — decidable
- connected — undecidable

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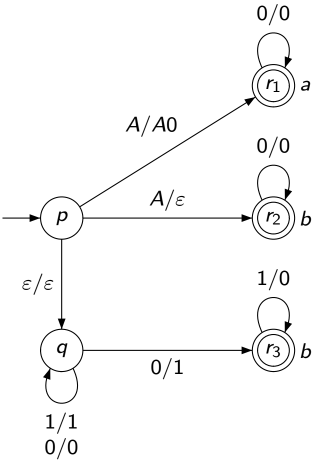

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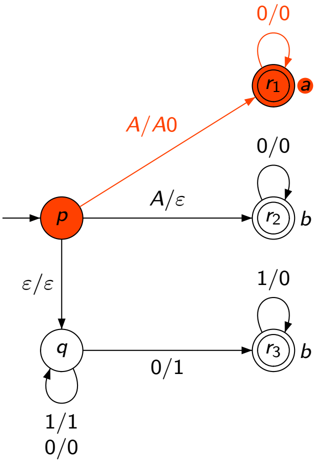

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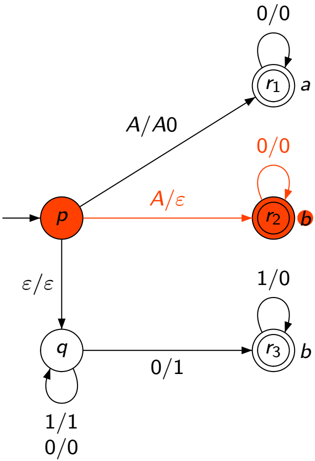

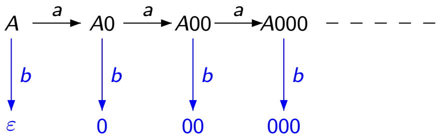

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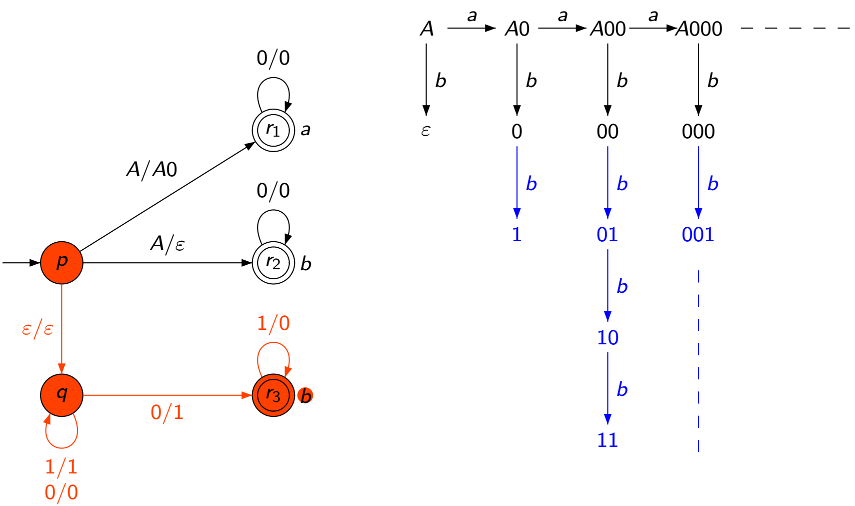

Properties of rational trees

Proposition

For rational trees

- *inclusion and equality is decidable*
- *accessibility (between vertices) is decidable*
- *effective rational set of leaves*
- *some sub-trees are non-rational*

Theorem (Carayol Morvan 06)

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The trace of a tree is the set of path labels from the root to the leaves

Conjecture

There are context-free languages which are not the trace of any rational tree

Example

The set of words having the same number of a's and b's seems not to be the trace of any rational tree

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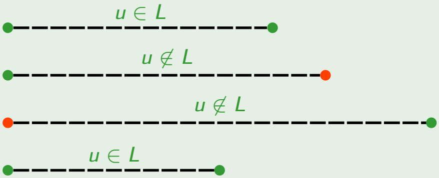
Given any context-sensitive language L we may construct such a rational graph (green and red states form rational sets)



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Given any context-sensitive language L (with alphabet X), there is a rational tree T labelled on $X \cup \{\#\}$ such that $Tr(T) \cap X^* = L$

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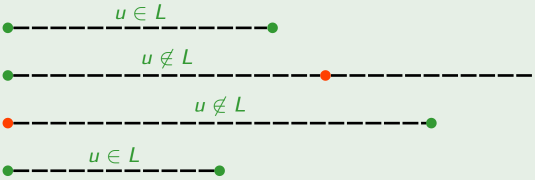

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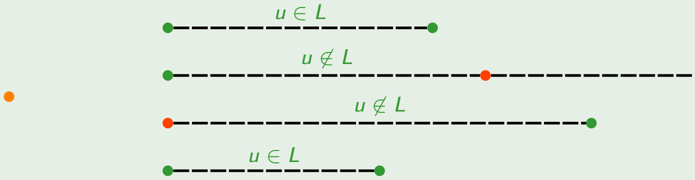

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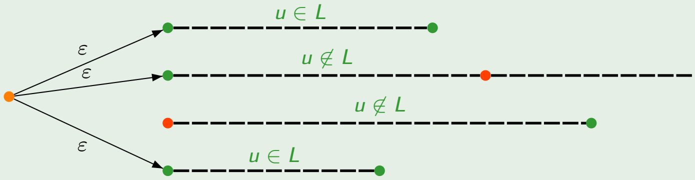

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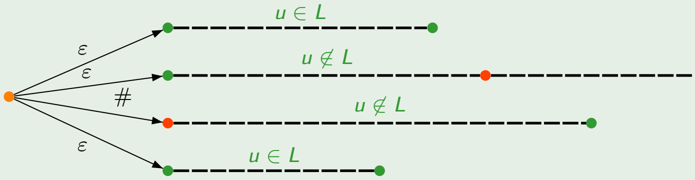

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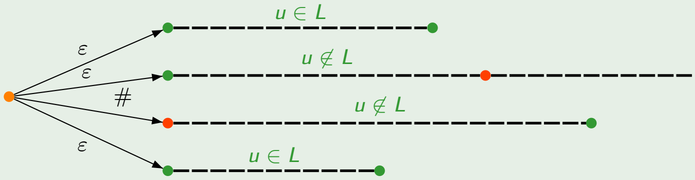

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It is undecidable to know if a rational graph is a forest

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The first order theory of rational forests is decidable

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The trace of rational forests between a root and leaves coincide with context-sensitive languages

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Rational Directed Acyclic Graphs

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A rational DAG is a rational graph with no cycle

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It is undecidable to know if a rational graph is a DAG

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There is a connected rational DAG with undecidable first-order theory

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Automatic Graphs

Definition

A graph over $X^* \times A \times X^*$ is automatic if it is recognized by a letter-to-letter transducer followed by a terminal function taking values in

$$(Rat(X^*) \times \{\varepsilon\}) \cup (\{\varepsilon\} \times Rat(X^*))$$

Example

The previous examples ($a^n b^n c^n$-graph and 2^n-tree) are automatic

Proposition (Hodgson 83)

The first order theory of automatic graphs is decidable

Theorem (Rispal 01)

The traces of automatic graphs are context-sensitive languages

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Monotonous Rational Graphs

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A rational graph is monotonous if for each transition u/v of its transducer either satisfy $|u| \leqslant |v|$ or $|u| \geqslant |v|$

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Accessibility is decidable for monotonous graphs

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Is the first order theory of monotonous graphs decidable?

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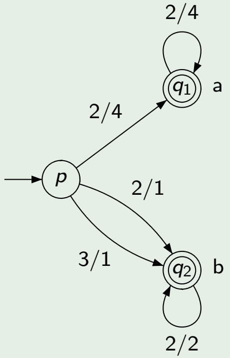

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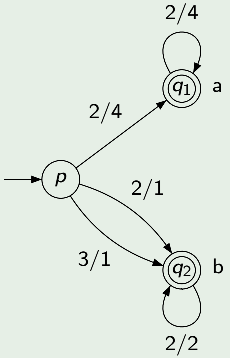

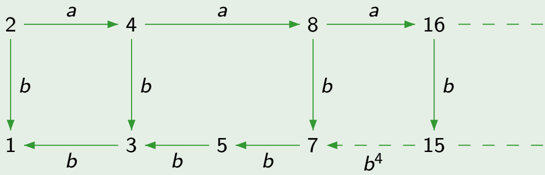

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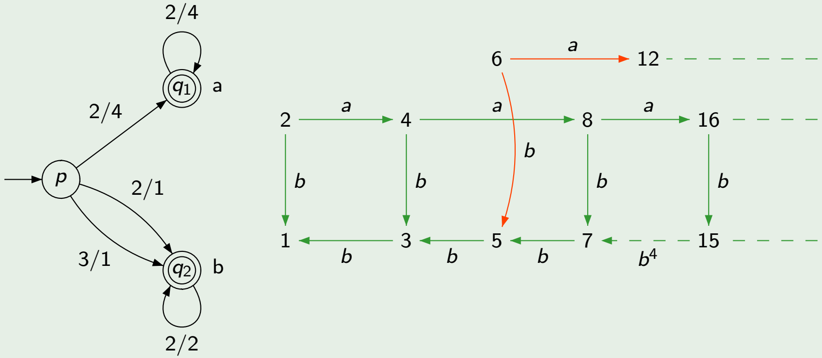

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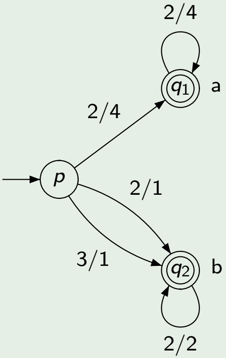

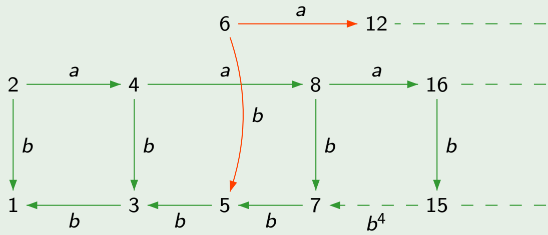

$$Tr(G, 2, 1) = \left\{ a^n b^{2^n} \mid n \in \mathbb{N} \right\} \notin CF(\{a, b\}^*)$$

Some results

Theorem (Eilenberg, Schützenberger 69)

The family of rational sets in a commutative monoid form a Boolean algebra

Corollary

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Several families of rational graphs

- Structural
  - Deterministic rational graphs
  - Rational trees
  - Rational DAG
- Automata driven

Future work

- Some families are promising
  Deterministic rational graphs, Monotonous rational graphs, Commutative rational graphs
- Exhibit a such a family with FO+ACC

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  - Automatic graphs
  - Monotonous rational graphs
  - Commutative rational graphs

Future work

- Some families are promising
  Deterministic rational graphs, Monotonous rational graphs, Commutative rational graphs
- Exhibit a such a family with FO+ACC decidable