# Dichotomies and Duality in First-order Model Checking Problems

Barnaby Martin

Department of Computer Science
University of Durham

Journées Montoises 2006, Rennes.

# Outline

Fix a logic $\mathcal{L}$. The *model checking problem* over $\mathcal{L}$ may be defined to have

- Input: a structure (model) $A$ and a sentence $\varphi$ of $\mathcal{L}$.
- Question: does $A \models \varphi$?

The complexity of this problem is sometimes known as the *combined complexity* of $\mathcal{L}$.

Fix a logic $\mathcal{L}$. The *model checking problem* over $\mathcal{L}$ may be defined to have

- Input: a structure (model) $A$ and a sentence $\varphi$ of $\mathcal{L}$.
- Question: does $A \models \varphi$?

The complexity of this problem is sometimes known as the *combined complexity* of $\mathcal{L}$.
This problem can be parameterised, either by the sentence $\varphi$, in which case the input is just $A$; or by the model $A$, in which case the input is just $\varphi$.
The maximal complexity of the problem parameterised by $\varphi$ is known as the *data complexity* of $\mathcal{L}$; the maximal complexity of the problem parameterised by $A$ is known as the *expression complexity* of $\mathcal{L}$.

Vardi has studied this problem, mostly for logics which subsume **FO**.

| logic | complexity | | |
|---|---|---|---|
| | data | expression | combined |
| q.f.**FO** | Logspace | Logspace | Logspace |
| **FO** | Logspace | Pspace | Pspace |
| TC | NLogspace | (N)Pspace | (N)Pspace |
| LFP | P | Exptime | Exptime |
| $\exists$**SO** | NP | NExptime | NExptime |

In all cases, these complexities are complete with respect to Logspace reductions. In most[1] of the cases it may be seen that the expression and combined complexities coincide, and are one exponential higher than the data complexity.

[1] Indeed, in all but the first. The first case is slightly anachronistic anyway since being Logspace-hard under Logspace reduction is trivial.

We will be interested only in logics $\mathcal{L}$ which are fragments of **FO**, and only in the parameterisation of their model checking problem by the model $A$. The fragments of **FO** which we consider derive from restricting which of the symbols of $\Gamma_1 := \{\neg, \land, \lor, \exists, \forall, =\}$ we allow. For example, we consider $\{\land, \exists\}$-**FO** to be that fragment of **FO** without negation, disjunction, universal quantification or equality.

We will be interested only in logics $\mathcal{L}$ which are fragments of **FO**, and only in the parameterisation of their model checking problem by the model $A$. The fragments of **FO** which we consider derive from restricting which of the symbols of $\Gamma_1 := \{\neg, \wedge, \vee, \exists, \forall, =\}$ we allow. For example, we consider $\{\wedge, \exists\}$-**FO** to be that fragment of **FO** without negation, disjunction, universal quantification or equality. For any $\Gamma \subseteq \Gamma_1$ we define the logic $\Gamma$-**FO** similarly, and we define the problem $\Gamma$-MC($A$) to have

- Input: a sentence $\varphi$ of $\Gamma$-**FO**.
- Question: does $A \models \varphi$?

The maximal complexity of this over all $A$ is therefore the expression complexity of $\Gamma$-**FO**.

The case for $\{\neg, \wedge, \vee, \exists, \forall, =\}$-**FO**, i.e. full first-order logic, is addressed by Vardi. It is known that $\{\neg, \wedge, \vee, \exists, \forall, =\}$-MC($A$), that is the problem

- Input: a sentence $\varphi$ of **FO**.
- Question: does $A \models \varphi$?

is Pspace-complete, if $||A|| > 1$; and in Logspace if $||A||$=1.

The case for $\{\neg, \wedge, \vee, \exists, \forall, =\}$-**FO**, i.e. full first-order logic, is addressed by Vardi. It is known that $\{\neg, \wedge, \vee, \exists, \forall, =\}$-MC($A$), that is the problem

- Input: a sentence $\varphi$ of **FO**.
- Question: does $A \models \varphi$?

is Pspace-complete, if $||A|| > 1$; and in Logspace if $||A||$=1. Pspace-hardness may be proved by reduction from *quantified satisfiability* (QSAT); Logspace membership comes via the (propositional) *boolean sentence value problem*.

The case for $\{\neg, \wedge, \vee, \exists, \forall, =\}$-**FO**, i.e. full first-order logic, is addressed by Vardi. It is known that $\{\neg, \wedge, \vee, \exists, \forall, =\}$-MC($A$), that is the problem

- Input: a sentence $\varphi$ of **FO**.
- Question: does $A \models \varphi$?

is Pspace-complete, if $||A|| > 1$; and in Logspace if $||A||$=1. Pspace-hardness may be proved by reduction from *quantified satisfiability* (QSAT); Logspace membership comes via the (propositional) *boolean sentence value problem*.
Similarly, it is known that $\{\neg, \wedge, \vee, \exists, \forall\}$-MC($A$) is Pspace-complete if $A$ contains any non-trivial relation (i.e. a relation that is non-empty and does not contain all tuples) and is in Logspace otherwise.

In this talk, we will be concerned with purely relational signatures and with those non-trivial positive fragments of **FO** which contain exactly one of the quantifiers. We have 12 cases to consider.

| Class I | Class II | Class III |
|---|---|---|
| $\{\vee, \exists\}$-**FO** | $\{\wedge, \exists\}$-**FO** | $\{\wedge, \vee, \exists\}$-**FO** |
| $\{\vee, \exists, =\}$-**FO** | $\{\wedge, \exists, =\}$-**FO** | $\{\wedge, \vee, \exists, =\}$-**FO** |
| $\{\wedge, \forall\}$-**FO** | $\{\vee, \forall\}$-**FO** | $\{\wedge, \vee, \forall\}$-**FO** |
| $\{\wedge, \forall, =\}$-**FO** | $\{\vee, \forall, =\}$-**FO** | $\{\wedge, \vee, \forall, =\}$-**FO** |

In this talk, we will be concerned with purely relational signatures and with those non-trivial positive fragments of **FO** which contain exactly one of the quantifiers. We have 12 cases to consider.

| Class I | Class II | Class III |
|---|---|---|
| $\{\vee, \exists\}$-**FO** | $\{\wedge, \exists\}$-**FO** | $\{\wedge, \vee, \exists\}$-**FO** |
| $\{\vee, \exists, =\}$-**FO** | $\{\wedge, \exists, =\}$-**FO** | $\{\wedge, \vee, \exists, =\}$-**FO** |
| $\{\wedge, \forall\}$-**FO** | $\{\vee, \forall\}$-**FO** | $\{\wedge, \vee, \forall\}$-**FO** |
| $\{\wedge, \forall, =\}$-**FO** | $\{\vee, \forall, =\}$-**FO** | $\{\wedge, \vee, \forall, =\}$-**FO** |

The model checking problem associated with the first logic of Class II, $\{\wedge, \exists\}$-**FO**, is essentially the *constaint satisfaction problem* (CSP).

The problem $\{\wedge, \exists\}$-MC($A$), that is the problem

- Input: a sentence $\varphi$ of $\{\wedge, \exists\}$-**FO**.
- Question: does $A \models \varphi$?

is better known as the *constraint satisfaction problem* CSP($A$).

[2] This conjecture was originally due to Feder and Vardi, although they gave no separating criterion. Bulatov, subsequently, conjectured a separating criterion.

The problem $\{\wedge, \exists\}$-MC($A$), that is the problem

- Input: a sentence $\varphi$ of $\{\wedge, \exists\}$-**FO**.
- Question: does $A \models \varphi$?

is better known as the *constraint satisfaction problem* CSP($A$). The question as to the precise complexity of this problem, over various $A$, has attracted much attention. The problem is always in NP and is often NP-complete, although cases in P are known. It is conjectured[2] that the problem is always either in P or NP-complete. This so-called CSP *dichotomy conjecture* remains open, but it has been settled for certain classes of model.

[2] This conjecture was originally due to Feder and Vardi, although they gave no separating criterion. Bulatov, subsequently, conjectured a separating criterion.

- (Schaefer 1978) Dichotomy when $||A|| \leq 2$ (i.e. on boolean models); recently extended to
- (Bulatov 2002) Dichotomy when $||A|| \leq 3$.
- (Hell/Nešetřil 1990) Dichotomy when $A$ ranges over undirected graphs.

  Specifically: if $A$ has a self-loop or is bipartite, then $\{\wedge, \exists\}$-MC($A$) is in P, otherwise it is NP-complete.

Whilst we can not answer the dichotomy conjecture in general, it provides the motivation for us to consider the analagous model checking problems for logics similar to $\{\vee, \exists\}$-**FO**.

But we have run ahead of ourselves. Let us consider the model checking problems associated with the logics of Class I: $\{\vee, \exists\}$-**FO**, $\{\vee, \exists, =\}$-**FO**, $\{\wedge, \forall\}$-**FO** and $\{\wedge, \forall, =\}$-**FO**.

But we have run ahead of ourselves. Let us consider the model checking problems associated with the logics of Class I: $\{\vee, \exists\}$-**FO**, $\{\vee, \exists, =\}$-**FO**, $\{\wedge, \forall\}$-**FO** and $\{\wedge, \forall, =\}$-**FO**.
It turns out that none of these model checking problems is particularly hard, indeed, they are all in Logspace. We will prove this for $\{\vee, \exists\}$-MC($A$) where $A$ is any digraph. We may consider any input to be prenex and of the form

$$\varphi \ := \ \exists \mathbf{v} \ E(v_1, v_1') \vee \ldots \vee E(v_m, v_m')$$

But to establish whether $A \models \varphi$, we need only cycle through $||A||^2$ pairs, looking for a witness to a disjunct.

We have already discussed $\{\wedge, \exists\}$-**FO** and its model checking problem – also known as the CSP. Owing to the rule of substitution, the logic $\{\wedge, \exists\}$-**FO** is very nearly as expressive as $\{\wedge, \exists, =\}$-**FO**. From the perspective of the complexity of their model checking problems, we consider them the same.

[3] That is, for each $R_i^{A}$, the relation $R_i^{\overline{A}}$ is defined by $\mathbf{x} \in R_i^{\overline{A}}$ iff $\mathbf{x} \notin R_i^{A}$.

We have already discussed $\{\wedge, \exists\}$-**FO** and its model checking problem – also known as the CSP. Owing to the rule of substitution, the logic $\{\wedge, \exists\}$-**FO** is very nearly as expressive as $\{\wedge, \exists, =\}$-**FO**. From the perspective of the complexity of their model checking problems, we consider them the same.

- For a model $A$, we define its complement $\overline{A}$ to have the same universe as $A$, but with relations which are the (set-theoretic) complements of the relations of $A$[3].

[3] That is, for each $R_i^A$, the relation $R_i^{\overline{A}}$ is defined by $\mathbf{x} \in R_i^{\overline{A}}$ iff $\mathbf{x} \notin R_i^A$.

We have already discussed $\{\wedge,\exists\}$-**FO** and its model checking problem – also known as the CSP. Owing to the rule of substitution, the logic $\{\wedge,\exists\}$-**FO** is very nearly as expressive as $\{\wedge,\exists,=\}$-**FO**. From the perspective of the complexity of their model checking problems, we consider them the same.

- For a model $A$, we define its complement $\overline{A}$ to have the same universe as $A$, but with relations which are the (set-theoretic) complements of the relations of $A$[3].

This brings us on to the logic $\{\vee,\forall\}$-**FO**, which is dual to $\{\wedge,\exists\}$-**FO** in the following sense.

[3]That is, for each $R_i^A$, the relation $R_i^{\overline{A}}$ is defined by $\mathbf{x} \in R_i^{\overline{A}}$ iff $\mathbf{x} \notin R_i^A$.

Consider a prenex sentence $\varphi$ of $\{\vee, \forall\}$-**FO**

$$\varphi \;:=\; \forall \mathbf{v}\; R_{\alpha_1}(\mathbf{v_1}) \vee \ldots \vee R_{\alpha_m}(\mathbf{v_m})$$

Now, $A \not\models \varphi$ iff

$$\begin{array}{llllll}
A & \not\models & & \forall \mathbf{v} & R_{\alpha_1}(\mathbf{v_1}) \vee \ldots \vee R_{\alpha_m}(\mathbf{v_m}) & \text{iff} \\
A & \not\models & \neg & \exists \mathbf{v} & \neg[R_{\alpha_1}(\mathbf{v_1}) \vee \ldots \vee R_{\alpha_m}(\mathbf{v_m})] & \text{iff} \\
A & \not\models & \neg & \exists \mathbf{v} & \neg R_{\alpha_1}(\mathbf{v_1}) \wedge \ldots \wedge \neg R_{\alpha_m}(\mathbf{v_m}) & \text{iff} \\
A & \models & & \exists \mathbf{v} & \neg R_{\alpha_1}(\mathbf{v_1}) \wedge \ldots \wedge \neg R_{\alpha_m}(\mathbf{v_m}) & \text{iff} \\
\overline{A} & \models & & \exists \mathbf{v} & R_{\alpha_1}(\mathbf{v_1}) \wedge \ldots \wedge R_{\alpha_m}(\mathbf{v_m}) & \text{iff}
\end{array}$$

$\overline{A} \models \varphi'$, where

$$\varphi' \;:=\; \exists \mathbf{v}\; R_{\alpha_1}(\mathbf{v_1}) \wedge \ldots \wedge R_{\alpha_m}(\mathbf{v_m}).$$

It follows that $\{\vee, \forall\}$-MC$(A)$ and *the complement of* $\{\wedge, \exists\}$-MC$(\overline{A})$ are equivalent under, e.g., Logspace reductions.

This reduction demonstrates that $\{\vee, \forall\}$-MC($A$) is always in co-NP, and that, if we choose some $A$ such that $\{\wedge, \exists\}$-MC($\overline{A}$) is NP-complete, then $\{\vee, \forall\}$-MC($A$) is co-NP-complete.
This tells us that the classification problem for $\{\vee, \forall\}$-MC($A$) is as hard as that for $\{\wedge, \exists\}$-MC($A$), and that a dichotomy holds for the former (between P and co-NP-complete) iff it holds for the latter (between P and NP-complete).

We can do better with the final logic of Class I, $\{\vee, \forall, =\}$-**FO**. This is not dual to the logic $\{\wedge, \exists, =\}$-**FO**, rather it is dual to the logic $\{\wedge, \exists\}$-**FO** when augmented with a *disequality* relation. Since a disequality relation on a structure is tantamount to a graph clique, we are immediately lead to the following.

- For structures $A$ such that $||A|| \geq 3$, the problem $\{\vee, \forall, =\}$-MC($A$) is co-NP-complete.

We can do better with the final logic of Class I, $\{\vee, \forall, =\}$-**FO**. This is not dual to the logic $\{\wedge, \exists, =\}$-**FO**, rather it is dual to the logic $\{\wedge, \exists\}$-**FO** when augmented with a *disequality* relation. Since a disequality relation on a structure is tantamount to a graph clique, we are immediately lead to the following.

- For structures $A$ such that $||A|| \geq 3$, the problem $\{\vee, \forall, =\}$-MC($A$) is co-NP-complete.

In fact, thanks to Schaefer's dichotomy on boolean models, we can go further, obtaining a complete classification.

We can do better with the final logic of Class I, $\{\vee, \forall, =\}$-**FO**. This is not dual to the logic $\{\wedge, \exists, =\}$-**FO**, rather it is dual to the logic $\{\wedge, \exists\}$-**FO** when augmented with a *disequality* relation. Since a disequality relation on a structure is tantamount to a graph clique, we are immediately lead to the following.

- For structures $A$ such that $||A|| \geq 3$, the problem $\{\vee, \forall, =\}$-MC($A$) is co-NP-complete.

In fact, thanks to Schaefer's dichotomy on boolean models, we can go further, obtaining a complete classification.

- For a model $A$, define $R_A$ to be the cartesian product of the non-empty relations of $A$.

We can do better with the final logic of Class I, $\{\vee, \forall, =\}$-**FO**. This is not dual to the logic $\{\wedge, \exists, =\}$-**FO**, rather it is dual to the logic $\{\wedge, \exists\}$-**FO** when augmented with a *disequality* relation. Since a disequality relation on a structure is tantamount to a graph clique, we are immediately lead to the following.

- For structures $A$ such that $||A|| \geq 3$, the problem $\{\vee, \forall, =\}$-MC$(A)$ is co-NP-complete.

In fact, thanks to Schaefer's dichotomy on boolean models, we can go further, obtaining a complete classification.

- For a model $A$, define $R_A$ to be the cartesian product of the non-empty relations of $A$.
- The $a$-ary relation $R \subseteq |A|^a$ is said to be $x$-valid, for some $x \in A$, iff $(x^a) = (x, \ldots, x) \in R$.

In full generality, the class of problems $\{\vee, \forall, =\}$-MC($A$) exhibits dichotomy, between those cases that are in P and those that are co-NP-complete. Specifically:

- If $||A|| = 1$, then the problem $\{\vee, \forall, =\}$-MC($A$) is in P.
- If $||A|| = 2$ then

  if $R_{\overline{A}}$ is 0-valid, 1-valid, horn, dual horn, bijunctive or affine, then $\{\vee, \forall, =\}$-MC($A$) is in P, otherwise $\{\vee, \forall, =\}$-MC($A$) is co-NP-complete.
- If $||A|| \geq 3$, then the problem $\{\vee, \forall, =\}$-MC($A$) is co-NP-complete.

The structures of the model checking problems $\{\wedge, \vee, \exists\}$-MC($A$) and $\{\wedge, \exists\}$-MC($A$) are somewhat similar in that both are always in NP and both are unique up to *homomorphism equivalence* of the template.

[4] Not just equivalent in some complexity-theoretic sense: by identical we mean that, for all $\varphi$ in $\{\wedge, \vee, \exists\}$-**FO**, $A \models \varphi$ iff $A' \models \varphi$.

The structures of the model checking problems $\{\wedge, \vee, \exists\}$-MC($A$) and $\{\wedge, \exists\}$-MC($A$) are somewhat similar in that both are always in NP and both are unique up to *homomorphism equivalence* of the template. By this we mean that $\{\wedge, \vee, \exists\}$-MC($A$) and $\{\wedge, \vee, \exists\}$-MC($A'$) are identical[4] iff $A$ and $A'$ are homomorphically equivalent, i.e. there exist homomorphisms both from $A$ to $A'$ and from $A'$ to $A$.

[4]Not just equivalent in some complexity-theoretic sense: by identical we mean that, for all $\varphi$ in $\{\wedge, \vee, \exists\}$-**FO**, $A \models \varphi$ iff $A' \models \varphi$.

The structures of the model checking problems $\{\wedge, \vee, \exists\}$-MC($A$) and $\{\wedge, \exists\}$-MC($A$) are somewhat similar in that both are always in NP and both are unique up to *homomorphism equivalence* of the template. By this we mean that $\{\wedge, \vee, \exists\}$-MC($A$) and $\{\wedge, \vee, \exists\}$-MC($A'$) are identical[4] iff $A$ and $A'$ are homomorphically equivalent, i.e. there exist homomorphisms both from $A$ to $A'$ and from $A'$ to $A$.
Through this observation and Hell and Nešetřil's CSP dichotomy for undirected graphs, we are able to derive that the class of problems $\{\wedge, \vee, \exists\}$-MC($A$) exhibits dichotomy. Specifically,

- if $R_A$ is either empty or $x$-valid, for some $x \in A$, then $\{\wedge, \vee, \exists\}$-MC($A$) is in Logspace, otherwise
- it $\{\wedge, \vee, \exists\}$-MC($A$) is NP-complete.

[4] Not just equivalent in some complexity-theoretic sense: by identical we mean that, for all $\varphi$ in $\{\wedge, \vee, \exists\}$-**FO**, $A \models \varphi$ iff $A' \models \varphi$.

$\{\wedge, \vee, \exists, =\}$-MC($A$) possesses the same dichotomy as $\{\wedge, \vee, \exists\}$-MC($A$) (although the possibility that $R_A$ is empty is removed).

$\{\wedge, \vee, \exists, =\}$-MC($A$) possesses the same dichotomy as $\{\wedge, \vee, \exists\}$-MC($A$) (although the possibility that $R_A$ is empty is removed).
The logics $\{\wedge, \vee, \forall\}$-**FO** and $\{\wedge, \vee, \exists\}$-**FO** are dual in the sense already described. It follows that the problems $\{\wedge, \vee, \forall\}$-MC($A$) exhibit dichotomy between those that are in Logspace and those that are co-NP-complete.

$\{\wedge, \vee, \exists, =\}$-MC($A$) possesses the same dichotomy as $\{\wedge, \vee, \exists\}$-MC($A$) (although the possibility that $R_A$ is empty is removed).
The logics $\{\wedge, \vee, \forall\}$-**FO** and $\{\wedge, \vee, \exists\}$-**FO** are dual in the sense already described. It follows that the problems $\{\wedge, \vee, \forall\}$-MC($A$) exhibit dichotomy between those that are in Logspace and those that are co-NP-complete.
Finally, the logic $\{\wedge, \vee, \forall, =\}$-**FO** is dual to the logic $\{\wedge, \vee, \exists\}$-**FO** augmented with disequality. It follows that the problems $\{\wedge, \vee, \forall, =\}$-MC($A$) exhibit a (slightly different) dichotomy, also between Logspace and co-NP-complete.

For certain fragments of **FO**, we have examined the complexity of the model checking problems in which the model acts as a parameter. In some cases we have obtained a simple classification – a dichotomy – in the other cases such a classification seems to be very hard.

For certain fragments of **FO**, we have examined the complexity of the model checking problems in which the model acts as a parameter. In some cases we have obtained a simple classification – a dichotomy – in the other cases such a classification seems to be very hard.

Ideally, as mathematicians, we would like to find problems that are neither too easy nor too hard.

For certain fragments of **FO**, we have examined the complexity of the model checking problems in which the model acts as a parameter. In some cases we have obtained a simple classification – a dichotomy – in the other cases such a classification seems to be very hard.

Ideally, as mathematicians, we would like to find problems that are neither too easy nor too hard.

A natural progression would be to consider those positive fragments of **FO** which contain both quantifiers. This leaves us with the following.

| Class IV | | Class V |
|---|---|---|
| $\{\wedge, \exists, \forall\}$-**FO** | $\{\vee, \exists, \forall\}$-**FO** | $\{\wedge, \vee, \exists, \forall\}$-**FO** |
| $\{\wedge, \exists, \forall, =\}$-**FO** | $\{\vee, \exists, \forall, =\}$-**FO** | $\{\wedge, \vee, \exists, \forall, =\}$-**FO** |

The model checking problem associated with $\{\wedge, \exists, \forall\}$-**FO** is essentially the *quantified* CSP. It is known that this problem may attain each of the complexities P, NP-complete and Pspace-complete, but no overarching classification is even conjectured. The remainder of Class IV are likely to be just as hard to classify[5].

[5] Possibly $\{\vee, \exists, \forall, =\}$-**FO** will be easier.

| Class IV | | Class V |
|---|---|---|
| $\{\wedge,\exists,\forall\}$-**FO** | $\{\vee,\exists,\forall\}$-**FO** | $\{\wedge,\vee,\exists,\forall\}$-**FO** |
| $\{\wedge,\exists,\forall,=\}$-**FO** | $\{\vee,\exists,\forall,=\}$-**FO** | $\{\wedge,\vee,\exists,\forall,=\}$-**FO** |

The model checking problem associated with $\{\wedge,\exists,\forall\}$-**FO** is essentially the *quantified* CSP. It is known that this problem may attain each of the complexities P, NP-complete and Pspace-complete, but no overarching classification is even conjectured. The remainder of Class IV are likely to be just as hard to classify[5].

Class V, with associated model checking complexities known to include each of P, NP-complete, co-NP-complete and Pspace-complete, may provide the richest area for future research.

[5] Possibly $\{\vee,\exists,\forall,=\}$-**FO** will be easier.