A Burnside Approach to the Termination of Mohri's Algorithm for Polynomially Ambiguous Min-Plus-Automata

Daniel Kirsten

Dresden University of Technology
Institut for Algebra

August 31, 2006

Definition:

An automaton $\mathcal{A}$ is called *polynomially ambiguous* if there exists some polynomial $P : \mathbb{N} \to \mathbb{N}$ such that for every $w \in \Sigma^*$ there are at most $P\big(|w|\big)$ accepting paths for w.

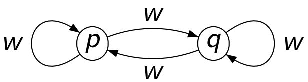
Definition:

An automaton $\mathcal{A}$ is called *polynomially ambiguous* if there exists some polynomial $P : \mathbb{N} \to \mathbb{N}$ such that for every $w \in \Sigma^*$ there are at most $P\big(|w|\big)$ accepting paths for w.

Theorem 1: Ibarra/Ravikumar 1986, Hromkovič/et al 2002

Let $\mathcal{A}$ be trim. The following assertions are equivalent:

- $\mathcal{A}$ is polynomially ambiguous.
- For every state q, every $w \in \Sigma^*$, we have $\left| q \overset{w}{\rightsquigarrow} q \right| \leq 1$.
- For every states p, q, every $w \in \Sigma^*$,



$$\Longrightarrow \qquad p = q.$$

Motivation:

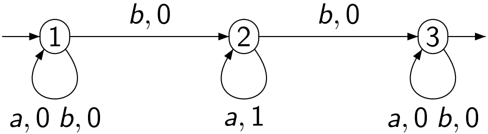
- less explored class of automata
- probably a large class of feasable WFA
- development of proof techniques

Motivation:

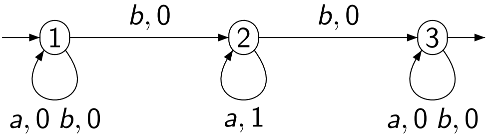
- less explored class of automata
- probably a large class of feasable WFA
- development of proof techniques
- they arise in the Cauchy-product of unambiguous/ finitely ambiguous series

$$(ST)(w) := \sum_{uv=w} S(u)T(v)$$

An Example:

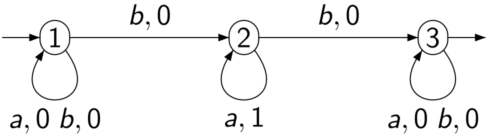


An Example:



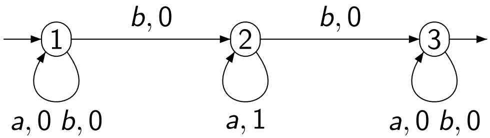
- $|\mathcal{A}|(w) \;=\; \min\left\{\, \ell \;\middle|\; ba^{\ell}b \text{ is a factor of } w \,\right\}$

An Example:



- $|\mathcal{A}|(w) \;=\; \min\left\{ \ell \;\middle|\; ba^{\ell}b \text{ is a factor of } w \right\}$
- $\mathcal{A}$ is polynomially ambiguous, $\quad \left|1 \overset{w}{\rightsquigarrow} 3\right| \;\leq\; |w|_b - 1 \;<\; |w|.$

An Example:



- $|\mathcal{A}|(w) = \min\left\{ \ell \;\middle|\; ba^{\ell}b \text{ is a factor of } w \right\}$
- $\mathcal{A}$ is polynomially ambiguous, $\left|1 \overset{w}{\rightsquigarrow} 3\right| \le |w|_b - 1 < |w|$.
- $|\mathcal{A}|$ is not the mapping of a finitely ambiguous WFA.

Mohri's Algorithm:

Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a pol. amb. WFA, i.e.,

- $Q = \{1, \ldots, n\}$ is a finite set,
- $\theta : \Sigma^* \to \mathbb{Z}^{Q \times Q}$ is a homomorphism,
- $\lambda, \varrho \in \mathbb{Z}^Q$.
- $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}, \qquad |\mathcal{A}|(w) := \lambda\, \theta(w)\, \varrho$

Mohri's Algorithm:

Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a pol. amb. WFA, i.e.,

- $Q = \{1, \ldots, n\}$ is a finite set,
- $\theta : \Sigma^* \to \mathbb{Z}^{Q \times Q}$ is a homomorphism,
- $\lambda, \varrho \in \mathbb{Z}^Q$.
- $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}, \qquad |\mathcal{A}|(w) := \lambda\,\theta(w)\,\varrho$

Let $B = (b_1, \ldots, b_n) \in \mathbb{Z}^Q$.

$\min(B) \ := \ \min\{b_i \mid i \in Q\}$

Mohri's Algorithm:

Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a pol. amb. WFA, i.e.,

- $Q = \{1, \ldots, n\}$ is a finite set,
- $\theta : \Sigma^* \to \mathbb{Z}^{Q \times Q}$ is a homomorphism,
- $\lambda, \varrho \in \mathbb{Z}^Q$.
- $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}, \qquad |\mathcal{A}|(w) := \lambda\, \theta(w)\, \varrho$

Let $B = (b_1, \ldots, b_n) \in \mathbb{Z}^Q$.

$\min(B) \;:=\; \min\{b_i \mid i \in Q\}$

$\mathrm{nf}(B) \;:=\; (-\min(B)) + B \;=\; \big(b_1 - \min(B), \; \ldots, \; b_n - \min(B)\big)$

Mohri's Algorithm:

Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a pol. amb. WFA, i.e.,

- $Q = \{1, \ldots, n\}$ is a finite set,
- $\theta : \Sigma^* \to \mathbb{Z}^{Q \times Q}$ is a homomorphism,
- $\lambda, \varrho \in \mathbb{Z}^Q$.
- $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}$, $\qquad |\mathcal{A}|(w) := \lambda\, \theta(w)\, \varrho$

Let $B = (b_1, \ldots, b_n) \in \mathbb{Z}^Q$.

$\min(B) \;:=\; \min\{b_i \mid i \in Q\}$

$\mathrm{nf}(B) \;:=\; (-\min(B)) + B \;=\; \big(b_1 - \min(B),\, \ldots,\, b_n - \min(B)\big)$

$\mathrm{nf}((1,2,3)) = (0,1,2) \qquad \mathrm{nf}((3,\infty,4)) = (0,\infty,1)$

$\mathrm{nf}((3,\infty,-4)) = (7,\infty,0)$

Let $Q' \subseteq \mathbb{Z}^Q$ be the least set which satisfies

- $\mathsf{nf}(\lambda) \in Q'$, and

Let $Q' \subseteq \mathbb{Z}^Q$ be the least set which satisfies

- $\mathsf{nf}(\lambda) \in Q'$, and
- for every $B \in Q'$, $a \in \Sigma$, $\quad \mathsf{nf}\big(B\theta(a)\big) \in Q'$.

Let $Q' \subseteq \mathbb{Z}^Q$ be the least set which satisfies

- $\mathsf{nf}(\lambda) \in Q'$, and
- for every $B \in Q'$, $a \in \Sigma$, $\mathsf{nf}\big(B\theta(a)\big) \in Q'$.

We have $Q' = \big\{\mathsf{nf}(\lambda\theta(w)) \,\big|\, w \in \Sigma^*\big\}$.

Let $Q' \subseteq \mathbb{Z}^Q$ be the least set which satisfies

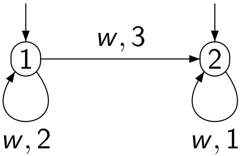
- $\mathrm{nf}(\lambda) \in Q'$, and
- for every $B \in Q'$, $a \in \Sigma$, $\mathrm{nf}\big(B\theta(a)\big) \in Q'$.

We have $Q' = \big\{\mathrm{nf}(\lambda\theta(w)) \,\big|\, w \in \Sigma^*\big\}$.

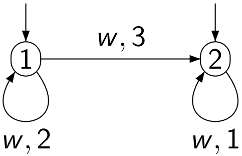
Mohri's Algorithm uses the set Q' as states.

It terminates iff Q' is finite.

An Example:

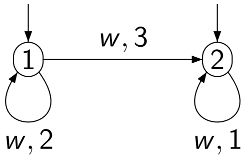


An Example:



For $k \geq 1$, we have $\lambda(\theta(w))^k = (2k, k)$ and

An Example:

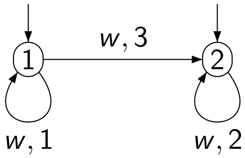


For $k \geq 1$, we have $\lambda(\theta(w))^k = (2k, k)$ and

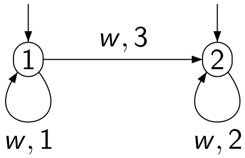
$\mathrm{nf}\big(\lambda(\theta(w))^k\big) = (k, 0)$, i.e.,

Mohri's algorithm does not terminate on the sequence $(w^k)_{k\geq 1}$.

Another Example:

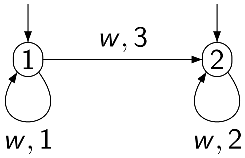


Another Example:



For $k \geq 2$, we have $\lambda(\theta(w))^k = (k, k+2)$ and

Another Example:



For $k \geq 2$, we have $\lambda(\theta(w))^k = (k, k+2)$ and

$\mathrm{nf}\big(\lambda(\theta(w))^k\big) = (0, 2)$, i.e.,

Mohri's algorithm terminates on the sequence $(w^k)_{k \geq 1}$.

Let $w \in \Sigma^*$ and $B = \theta(w)$.

Assume that B has an idempotent structure, i.e.,

$$B[i,j] \neq \infty \iff \big(BB\big)[i,j] \neq \infty \quad \text{for all } i,j \in Q.$$

Let $w \in \Sigma^*$ and $B = \theta(w)$.

Assume that B has an idempotent structure, i.e.,

$$B[i,j] \neq \infty \iff \big(BB\big)[i,j] \neq \infty \qquad \text{for all } i,j \in Q.$$

For $i,j \in Q$ let $i \leq_B j$ iff $B[i,j] \neq \infty$.

Let $w \in \Sigma^*$ and $B = \theta(w)$.

Assume that B has an idempotent structure, i.e.,

$$B[i,j] \neq \infty \iff \big(BB\big)[i,j] \neq \infty \qquad \text{for all } i,j \in Q.$$

For $i,j \in Q$ let $i \leq_B j$ iff $B[i,j] \neq \infty$.

The relation $\leq_B$ is transitive and antisymmetric,

Let $w \in \Sigma^*$ and $B = \theta(w)$.

Assume that B has an idempotent structure, i.e.,

$$B[i,j] \neq \infty \iff \big(BB\big)[i,j] \neq \infty \qquad \text{for all } i,j \in Q.$$

For $i,j \in Q$ let $i \leq_B j$ iff $B[i,j] \neq \infty$.

The relation $\leq_B$ is transitive and antisymmetric,

but not necessarily reflexive of irreflexive,

i.e., $\leq_B$ is almost a partial ordering.

A subset $C \subseteq Q$ is a *clone* iff

there exists some $v \in \Sigma^*$ such that $C = \left\{ i \in Q \,\middle|\, \lambda\theta(v)[i] \neq \infty \right\}$.

A subset $C \subseteq Q$ is a *clone* iff

there exists some $v \in \Sigma^*$ such that $C = \{i \in Q \mid \lambda\theta(v)[i] \neq \infty\}$.

C and B are *stable* iff $C = \{j \in Q \mid B[i,j] \neq \infty \text{ for some } i \in C\}$.

A subset $C \subseteq Q$ is a *clone* iff

there exists some $v \in \Sigma^*$ such that $C = \{i \in Q \mid \lambda\theta(v)[i] \neq \infty\}$.

C and B are *stable* iff $C = \{j \in Q \mid B[i,j] \neq \infty \text{ for some } i \in C\}$.

C and B satisfy the *clones property* if

for every $i \in C$ which is minimal w.r.t. $\leq_B$,

the value $B[i,i]$ is minimal among $B[j,j]$ for $j \in C$.

A subset $C \subseteq Q$ is a *clone* iff

there exists some $v \in \Sigma^*$ such that $C = \{ i \in Q \mid \lambda\theta(v)[i] \neq \infty \}$.

C and B are *stable* iff $C = \{ j \in Q \mid B[i,j] \neq \infty \text{ for some } i \in C \}$.

C and B satisfy the *clones property* if

for every $i \in C$ which is minimal w.r.t. $\leq_B$,

the value $B[i,i]$ is minimal among $B[j,j]$ for $j \in C$.

Lemma:
The set $\{ \mathrm{nf}(\lambda\theta(vw^k)) \mid k \in \mathbb{N} \}$ is finite **iff**

C and B satisfy the clones property.

A pol. amb. WFA $\mathcal{A}$ satisfies the *clones property* if

- for every clone C,

A pol. amb. WFA $\mathcal{A}$ satisfies the *clones property* if

- for every clone C,
- for every $w \in \Sigma^*$ such that $B := \theta(w)$ has an idempotent structure and C and B are stable,

A pol. amb. WFA $\mathcal{A}$ satisfies the *clones property* if

- for every clone C,
- for every $w \in \Sigma^*$ such that $B := \theta(w)$ has an idempotent structure and C and B are stable,

C and B satisfy the clones property.

A pol. amb. WFA $\mathcal{A}$ satisfies the *clones property* if

- for every clone C,
- for every $w \in \Sigma^*$ such that $B := \theta(w)$ has an idempotent structure and C and B are stable,

C and B satisfy the clones property.

Theorem 2: Kirsten 2005

Let $\mathcal{A}$ be trim, polynomially ambiguous WFA. The following assertions are equivalent:

1. Mohri's algorithm terminates on $\mathcal{A}$.

A pol. amb. WFA $\mathcal{A}$ satisfies the *clones property* if

- for every clone C,
- for every $w \in \Sigma^*$ such that $B := \theta(w)$ has an idempotent structure and C and B are stable,

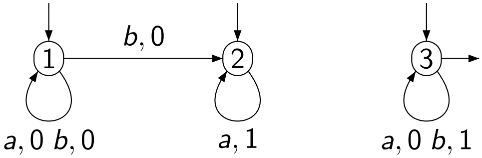
C and B satisfy the clones property.

Theorem 2: Kirsten 2005

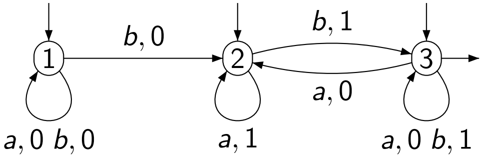
Let $\mathcal{A}$ be trim, polynomially ambiguous WFA. The following assertions are equivalent:

1. Mohri's algorithm terminates on $\mathcal{A}$.
2. For every $v, w \in \Sigma^*$, Mohri's algorithm terminates on the sequence $(vw^k)_{k \geq 1}$ on $\mathcal{A}$.
3. The WFA $\mathcal{A}$ satisfies the clones property.

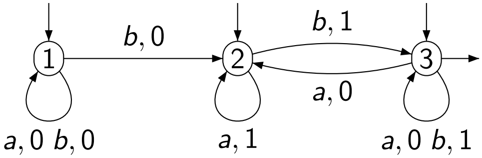
A bad Example:



A bad Example:

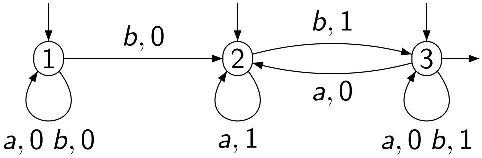


A bad Example:



- For every $v, w \in \Sigma^*$, Mohri's algorithm terminates on $(vw^k)_{k \geq 1}$.

A bad Example:



- For every $v, w \in \Sigma^*$, Mohri's algorithm terminates on $(vw^k)_{k\geq 1}$.
- Mohri's algorithm does not terminate on $baba^2ba^3ba^4b\ldots$

(2)$\Rightarrow$(1) in Theorem 2 does not hold for $\mathcal{A}$.