Tomi Kärki

Compatibility relations on codes and free monoids

University of Turku and Turku Centre for Computer Science

(TUCS)

Introduction

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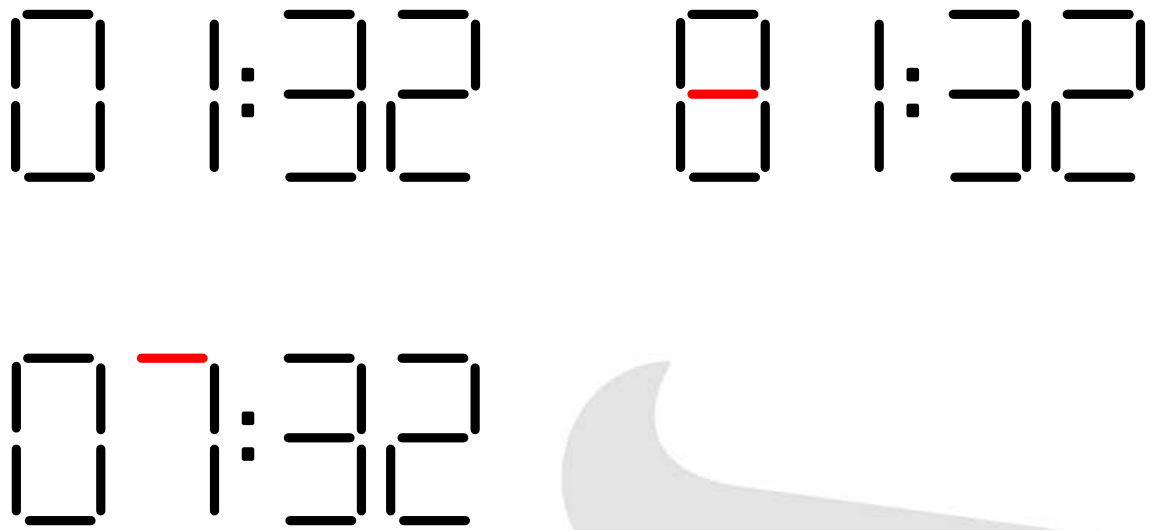

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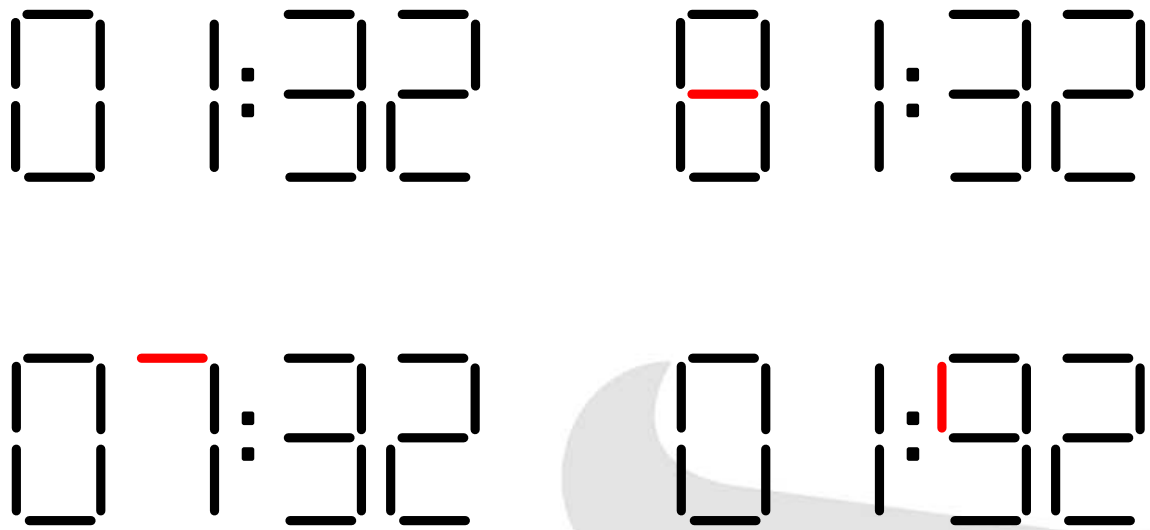

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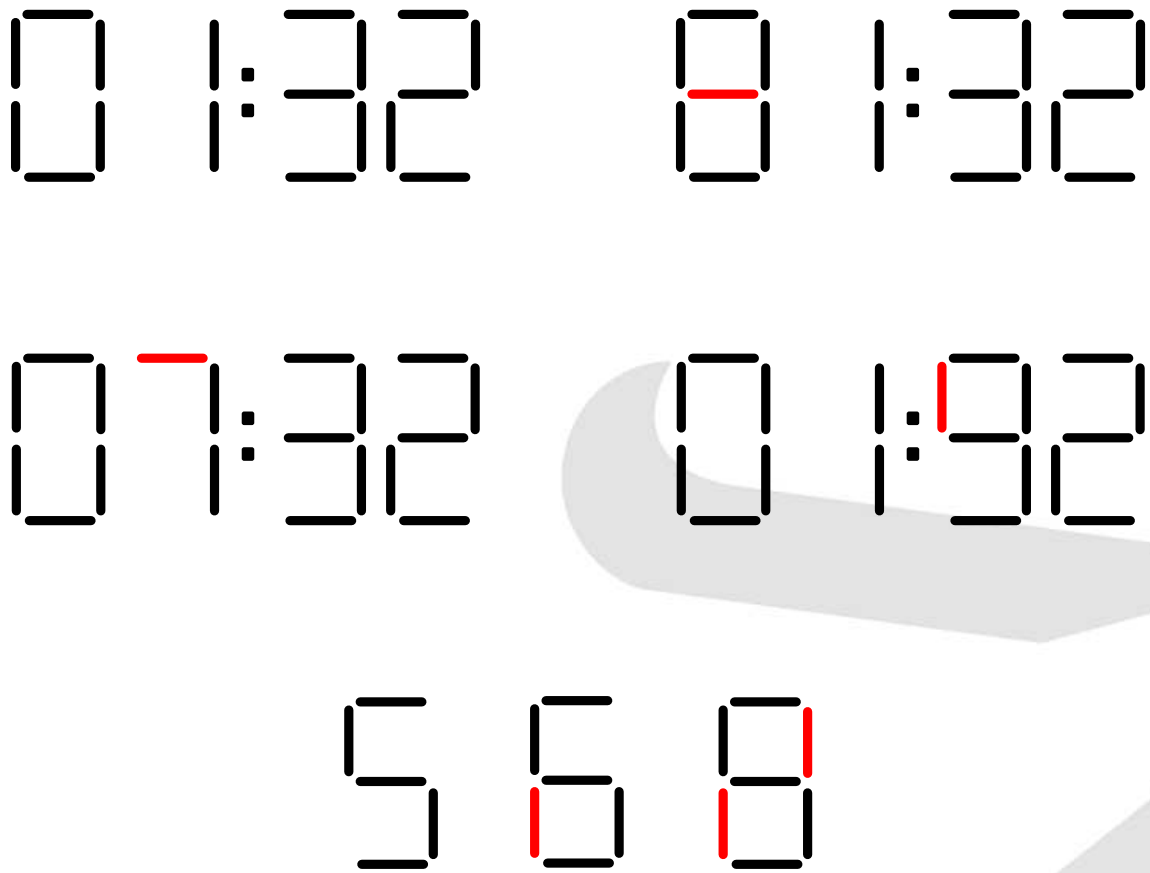

Outline of Topics

- Word relations
- Relational codes
- Minimal and maximal relations
- Relationally free monoids and stability
- Hulls
- Defect effect

Notations

A	an alphabet
ε	empty word
X	a set of words over A^*
$R \subseteq X \times X$	a relation on X
$x\,R\,y$	$(x, y) \in R$
ι_X	$\{(x, x) \mid x \in X\}$
Ω_X	$\{(x, y) \mid x, y \in X\}$
$\langle R \rangle$	reflexive and symmetric closure of R
R_Y	$R \cap (Y \times Y)$
$R(X)$	$\{u \in A^* \mid \exists\, x \in X :\ x\,R\,u\}$

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- If $u \, R \, v$, then words u and v are *R-compatible*
- $\begin{cases} \text{multiplicativity:} & u \, R \, v, u' \, R \, v' \Rightarrow uu' \, R \, vv', \\ \text{simplifiability:} & uu' \, R \, vv', |u| = |v| \Rightarrow u \, R \, v, \; u' \, R \, v' \end{cases}$

Word relations

Example 1. $A = \{a, b, c\}$
$R = \langle\{(a, b)\}\rangle = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$

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$$R_{\uparrow} = \langle\{(\diamondsuit, a) \mid a \in A\}\rangle$$

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- (R, S)-code *relational* code
  (R, ι)-code *strong* R-code
  (R, R)-code *weak* R-code
  (ι, ι)-code code

Relational codes

Example. $A = \{a, b, c\}$
$X = \{ab, c\}$
$S = \iota$

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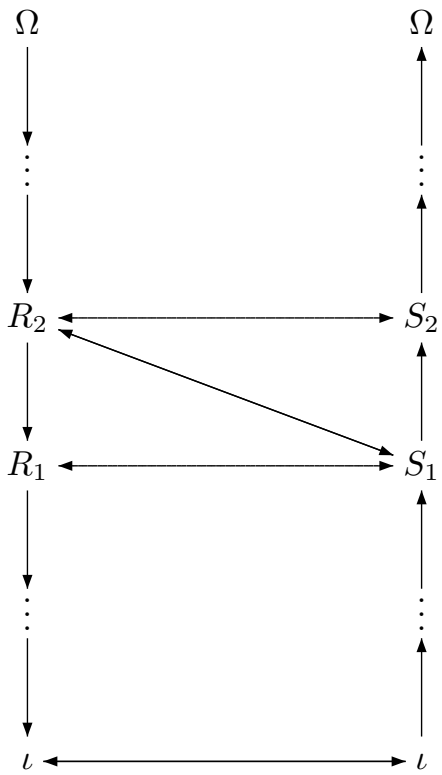

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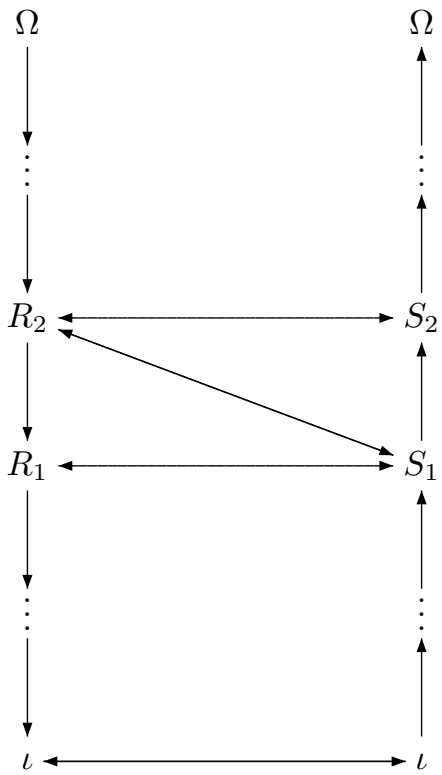

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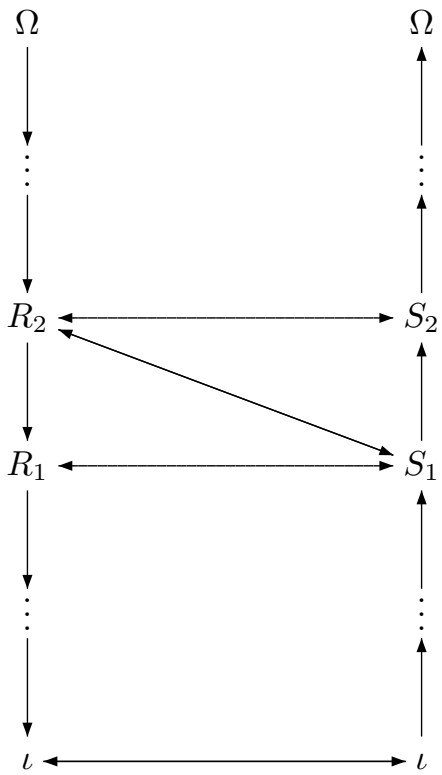

Theorem 3. Every (R, S)-code X is a code.

Theorem 4. Let X be a subset of A^*. X is an (R, S)-code $\Leftrightarrow$ X is an (R, R)-code and $R_X \subseteq S_X$.

Modified Sardinas-Patterson algorithm

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- Let $i \geq 2$ satisfy $U_i = U_{i-t}$ for some $t > 0$
- X is a weak R-code if and only if

$$\boxed{\varepsilon \notin \bigcup_{j=1}^{i-1} U_j}$$

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$\Longrightarrow$ X is not an (R, R)-code

$$ca.ca \, R \, c.abb$$

Minimal and maximal relations

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Instance:	$X \subseteq A^+$, relation S, $k \in \mathbb{N}$
Question:	Is max. size of $R \in R_{\max}(X, S) \geq k$?

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NP-complete

Relationally free monoids

A monoid $M \subseteq A^*$ is (R, S)-*free* if it has a subset $B \subseteq M$ (called an (R, S)-base of M) such that

$$(i) \quad M = B^*,$$

$$(ii) \quad B \text{ is an } (R, S)\text{-code.}$$

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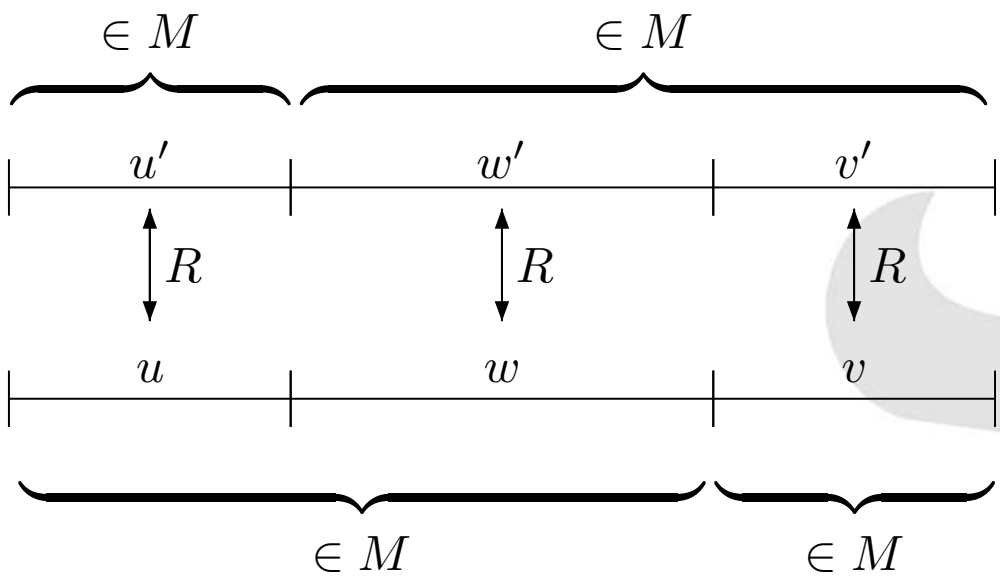
Theorem 5. X is (R, S)-code $\Leftrightarrow$ X^* is (R, S)-free with minimal generating set X

Theorem 6. M is (R, S)-free $\Leftrightarrow$ M is (R, R)-free and $R_B \subseteq S_B$ for the base B

Stability

A monoid $M \subseteq A^*$ is (R, S)-*stable* if $\forall u, v, w, u', v', w' \in A^*$:



$$\Rightarrow u, w \in M, u \, S \, u'$$

Stability

Theorem 7 (Generalized Schützenberger's criterium).

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Theorem 8 (Generalized Tilson's result). Any nonempty intersection of (R,S)-free monoids of A^* is (R,S)-free.

Hulls

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- $F_{(R,S)}(X)$ is the (R,S)-*free hull* of X
- **Theorem 9.** Let $F_R = F_{(R,R)}(X)$.
  $F_{(R,S)}(X)$ exists $\Leftrightarrow R_{F_R} \subseteq S_{F_R}$. Then $F_{(R,S)}(X) = F_R$.

Hulls

$$C_f(X) = \{(u,v) \in X \times X \mid (u,v) \notin R, uX^* \cap R(vX^*) \neq \emptyset\}.$$

Algorithm 1 (*Base of (R,R)-free hull A_f*). *Input: finite $X \subseteq A^+$. Set $X_0 = X$, and iterate for $j \geq 0$.*

1. *Choose $(u,v) \in C_f(X_j, R)$ such that $u = u'u''$, where $|u'| = |v|$ and $u'' \in A^+$. If no such pair exists, then stop and return $A_f(X) = X_j$.*
2. *Set $R'(u) = \{\mathsf{pref}_{|u'|}(w) \mid w \in (R_{X_j})^+(u)\}$ and set $R''(u) = \{\mathsf{suf}_{|u''|}(w) \mid w \in (R_{X_j})^+(u)\}$, where $(R_{X_j})^+$ is the transitive closure of R_{X_j}.*
3. *Set $X_{j+1} = \big(X_j \setminus (R_{X_j})^+(u)\big) \cup R'(u) \cup R''(u)$.*

Defect effect

Theorem (Defect theorem). Let $X \subseteq A^+$ be a finite set and let B be the base of the free hull of X. Then $|B| \leq |X|$, and the equality holds if and only if X is a code.

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- $G_R(X) = (V, E)$: $V = X$, $(u, v) \in E \Leftrightarrow u\,R\,v$
- $c(X, R) =$ the number of connected components of $G_R(X)$.

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- $c(X, R) =$ the number of connected components of $G_R(X)$.

Theorem 11 (Generalized defect theorem). Let X be a finite subset of A^* and let B be the base of the (R, R)-free hull of X. Then $c(B, R) \leq c(X, R)$, and the equality holds if and only if X is an (R, R)-code.

Defect effect

- *pcodes*: $(R_\uparrow, \iota)$-codes over $A_\diamondsuit$

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Corollary 1 (Defect theorem of partial words). Let X be a finite set of partial words, i.e., a set of words over the alphabet $A_\diamondsuit$. Suppose that pfree hull of X exists and let B be its base. Then $|B| \leq |X|$, and the equality holds if and only if X is a pcode.

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