On testing the equality of sofic systems

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Definitions

Given a finite automaton $\mathcal{A} = (Q, \Sigma, \delta)$

- $w \in \Sigma^*$ is a finite word,
- $u \in {}^{\omega}\Sigma$, $v \in \Sigma^{\omega}$ and $w \in {}^{\omega}\Sigma^{\omega}$ are left, right and bi-infinite words,

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- $(q_i)_{1 \le i \le k+1}$ is a finite path in $\mathcal{A}$ labeled by $w = (a_i)_{1 \le i \le k}$ iff $q_{i+1} \in \delta(q_i, a_i)$,
- $(q_i)_{i \le 0}$, $(q_i)_{i \ge 0}$ and $(q_i)_{i \in Z}$ are left, right and bi-infinite paths labeled by $(a_i)_{i<0}$, $(a_i)_{i \ge 0}$ and $(a_i)_{i \in Z}$ iff $q_{i+1} \in \delta(q_i, a_i)$.

Local Automata

A finite automaton $\mathcal{A} = (Q, \Sigma, \delta)$ is called *local* if there exist values *m* and *k*, $0 \leq m \leq k$, such that any two equally labeled paths of length *k* go through the same state at time *m*.

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The smallest value for *k* is called the synchronization delay of $\mathcal{A}$.

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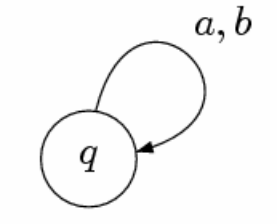
  - **Obs**. We may restrict ourselves to the non-transient part of the automaton (all states are part of some bi-infinite path)

$\omega\omega$Automata

Example:



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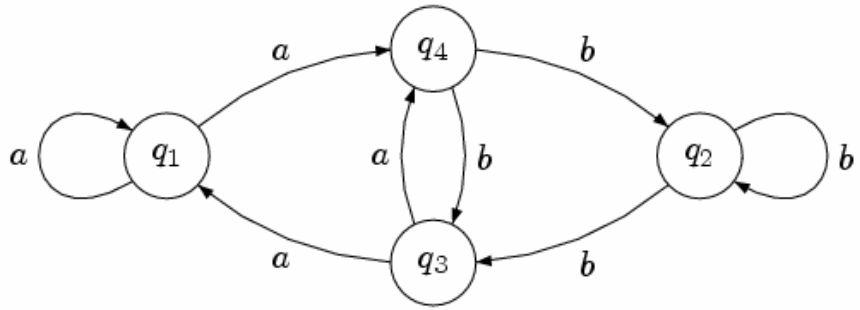

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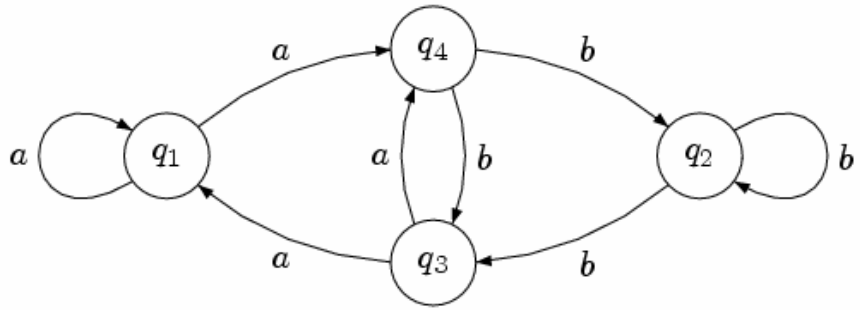

An automaton is $\omega\omega$ injective iff it is local. Moreover, the class of $\omega\omega$ bijective automata is strictly included in the class of local automata.

Decision algorithms

Given an automaton $\mathcal{A} = (Q, \Sigma, \delta)$ how efficiently can we decide whether:

- $\mathcal{A}$ is $\omega\omega$ injective
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  - Given two $\omega\omega$ injective automata, decide whether the two languages of bi-infinite words obtained as labels of bi-infinite paths are equal, i.e. the two automata are $\omega\omega$ equivalent.

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- Any subshift obtained in this way is called a sofic system, while the automaton generating it is called a presentation.
- **Obs**. A sofic system may have more than one presentation.

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A subshift is of finite type iff it has a local presentation

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  - Decide whether two subshifts of finite type are equal, given their presentation as $\omega\omega$ injective (local) automata.

Decision algorithms- $\omega\omega$injectivity

Quadratic in the number of states of the automaton (M.-P. Beal, Codage Symbolique, 1993)

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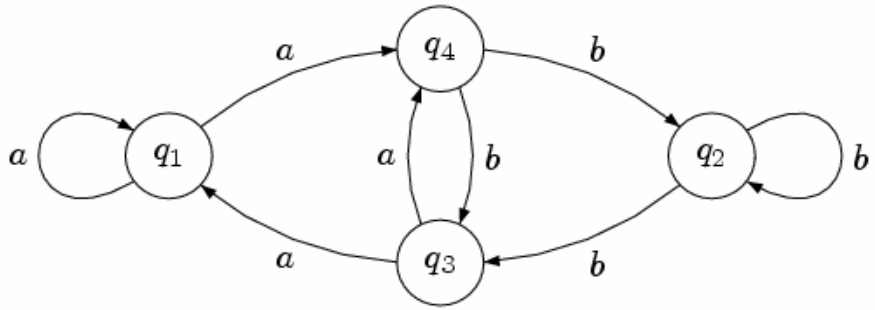

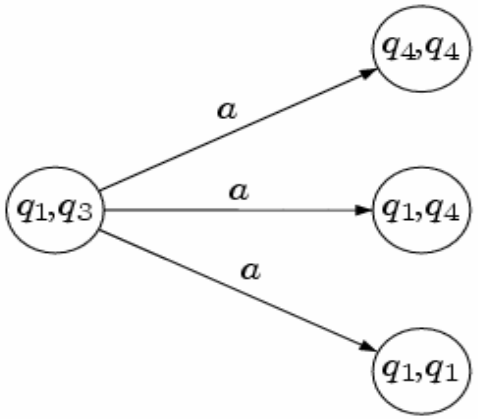

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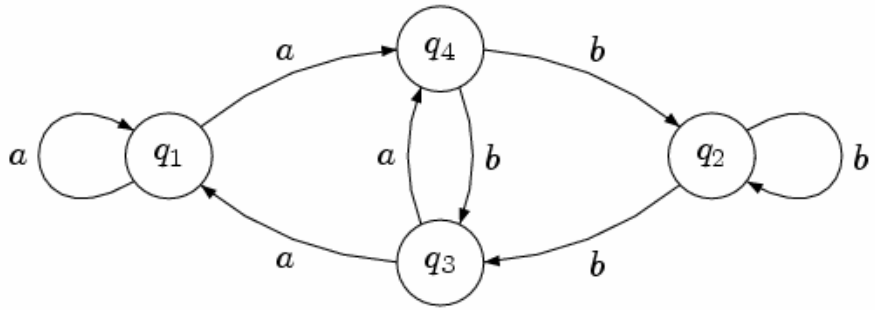

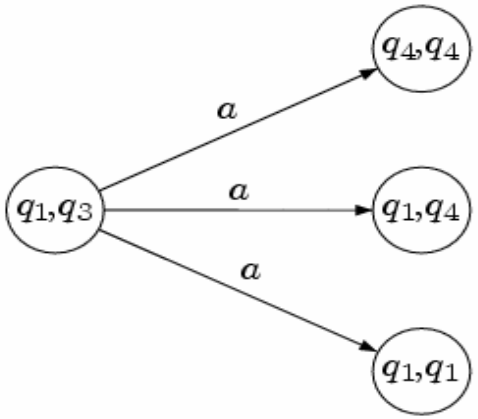

- Test for the existence of a loop containing a vertex of the form (q_i, q_j) , with $q_i \neq q_j$.

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- Given a finite automaton $\mathcal{A} = (Q, \Sigma, I, \delta, F)$, we construct $\mathcal{A}' = (Q, \Sigma \cup \{\#\}, \delta')$ such that for all $q_f \in F$ and $q_s \in I$, we add a transition $q_s \in \delta'(q_f, \#)$.

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- Testing whether a finite automaton is $\omega\omega$surjective is similar to testing whether it recognize the language Σ^* , when all the states are both initial and final.

Decision algorithms- $\omega\omega$ equivalence

Testing whether two automata are $\omega\omega$ equivalent can be done using a polynomial time algorithm.

Welch sets

Given a finite automaton $\mathcal{A} = (Q, \Sigma, \delta)$

- For any word $w \in \Sigma^{\omega}$ define the *left Welch set* of *w* as the set of states from which a path labeled by *w* can start.

$$L(w) = \{q \in Q \mid (q, q_1, q_2, \ldots) \text{ is a path in } \mathcal{A} \text{ labeled by } w\}$$

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- For any word $w \in {}^{\omega}\Sigma$ define the *right Welch set* of *w* as the set of states in which a path labeled by *w* can stop.

$$R(w) = \{q \in Q \mid (\ldots, q_{-2}, q_{-1}, q) \text{ is a path in } \mathcal{A} \text{ labeled by } w\}$$

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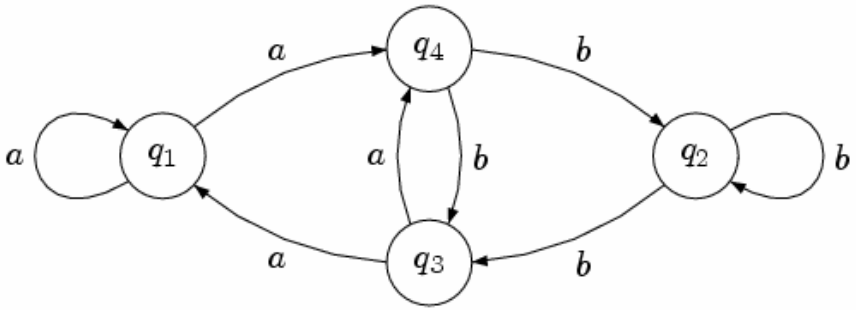

$$L(a^{\omega}) = \{q_1, q_3\}$$

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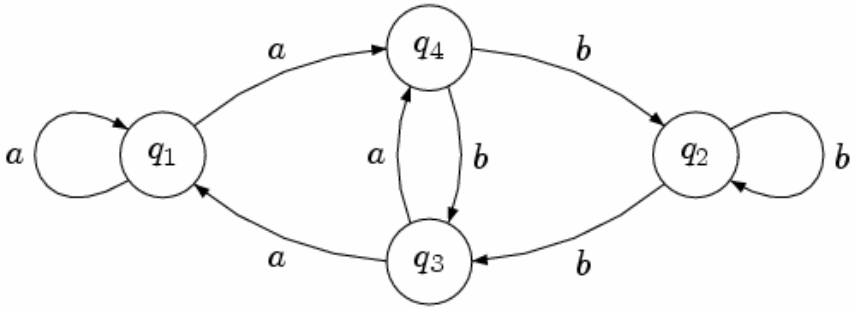

$$L(a^{\omega}) = \{q_1, q_3\} \qquad \vec{L} = (1, 0, 1, 0)$$

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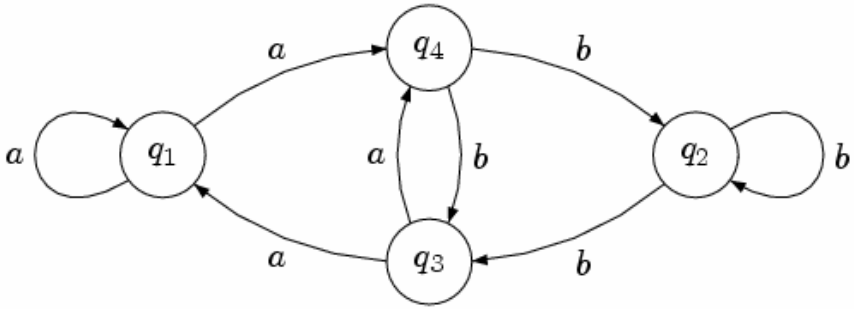

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- Given an $\omega\omega$injective automaton $\mathcal{A}$, if $\vec{L}_1$ *and* $\vec{R}_1$ *are two left/right Welch vectors, any other Welch vector can be obtained by applying some suitable linear functions.*

Extended Welch vectors

Given two finite autonata $\mathcal{A} = (Q, \Sigma, \delta)$ and $\mathcal{A}' = (Q', \Sigma, \delta')$

- For any word $w \in \Sigma^{\omega}$, based on the two left Welch sets *L* and *L'* corresponding to *w* on both automata, we define the *extended left Welch vector* of *w* as:

$$\vec{L}^{cc}(w) = \left(-\vec{L}, \vec{L}'\right)$$

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- For any word $w \in {}^{\omega}\Sigma$, based on the two right Welch sets *R* and *R'* corresponding to *w* on both automata, we define the *extended right Welch vector* of *w* as:

$$\vec{R}^{cc}(w) = \left(\vec{R}, \vec{R}'\right)$$

Decision algorithms- $\omega\omega$ equivalence

Two $\omega\omega$ injective automata are $\omega\omega$ equivalent iff for every $w \in \Sigma^{\omega}$ and $v \in {}^{\omega}\Sigma$

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We would like to test the above property only on some "representative" Welch vectors, i.e., a set of linear independent left/right Welch vectors.

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Given two $\omega\omega$ injective automata $\mathcal{A}$ and $\mathcal{A}'$ with n, respectively n' states, if $\{\vec{L}_1^{cc}, \ldots, \vec{L}_{n+n'}^{cc}\}$ is a <u>well selected</u> set of $n+n'$ extended left Welch vectors, then:

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- <u>either</u> *any other extended left Welch vector can be obtained by applying some suitable linear functions,*
- <u>*or*</u> *there exists* $u \in \Sigma^*$ *and* $1 \leq i \leq n+n'$ *such that* $h_u(\vec{L}_i^{cc}) = (-\vec{L}, \vec{0})$ *or* $h_u(\vec{L}_i^{cc}) = (\vec{0}, \vec{L}')$.

($\mathcal{A}$ *and* $\mathcal{A}'$ *are not* $\omega\omega$ *equivalent) .*

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Based on these *n+n'* vectors we compute a basis $\{\vec{L}_1^{cc}, \ldots, \vec{L}_k^{cc}\}$ of the vector space:

$$\langle \{h_u(\vec{L}_i^{cc}) \mid u \in \Sigma^*,\ 1 \leq i \leq n+n'\} \rangle$$

Decision algorithms- $\omega\omega$equivalence

Similarly, we obtain a basis $\{\vec{R}_1^{cc}, \ldots, \vec{R}_l^{cc}\}$ for

$$\langle \{ g_u(\vec{R}_i^{cc}) \mid u \in \Sigma^*,\ 1 \leq i \leq n + n' \} \rangle$$

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The two $\omega\omega$injective automata are $\omega\omega$equivalent iff for all $1 \leq i \leq k$ and $1 \leq j \leq l$, $\vec{L}_i^{cc} \cdot \vec{R}_j^{cc} = 0$.

Decision algorithms- $\omega\omega$equivalence

The Algorithm:

Decision algorithms- $\omega\omega$equivalence

The Algorithm:

- Compute the sets $\{\vec{L}_1^{cc}, \ldots, \vec{L}_{n+n'}^{cc}\}$ and $\{\vec{R}_1^{cc}, \ldots, \vec{R}_{n+n'}^{cc}\}$;

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The Algorithm:

- Compute the sets $\{\vec{L}_1^{cc}, \ldots, \vec{L}_{n+n'}^{cc}\}$ and $\{\vec{R}_1^{cc}, \ldots, \vec{R}_{n+n'}^{cc}\}$;
- Compute the basses $\{\vec{L}_1^{cc}, \ldots, \vec{L}_k^{cc}\}$ and $\{\vec{R}_1^{cc}, \ldots, \vec{R}_l^{cc}\}$;

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- Compute the basses $\{\vec{L}_1^{cc}, \ldots, \vec{L}_k^{cc}\}$ and $\{\vec{R}_1^{cc}, \ldots, \vec{R}_l^{cc}\}$;
- Test whether for all $1 \leq i \leq k$ and $1 \leq j \leq l$, $\vec{L}_i^{cc} \cdot \vec{R}_j^{cc} = 0$.

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- Decide whether two subshifts of finite type are equal, given their presentation as $^{\omega\omega}$injective automata. (polynomial time complexity – here)

- Does there exists a polynomial time algorithm deciding whether two subshifts of finite type are equal, given an arbitrary presentations for these subshifts, i.e two (not necessarily local) finite automata?

Thank you !