11th Mons Days
of Theoretical Computer Science
30th August – 2nd September, 2006, Rennes

Jānis Buls
University of Latvia

FROM BI-IDEALS TO PERIODICITY

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- $A^\infty = A^* \cup A^\omega$

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- Here $|u|$ — the lenght of word u;
- $\mathrm{Pref}(x)$ — the set of all prefixes of x.

- A sequence of words $\quad v_0, v_1, \ldots, v_n, \ldots$

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♣ A word is recurrent if and only if it is a bi-ideal.

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then a bi-ideal generated by $(u_0, u_1, \ldots, u_{m-1})$

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- Let $A = \{0, 1\}$ and $m = n = 10$

  then probability $p \leq \dfrac{1}{2^{100}}$.

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namely, these sets contain the words with different size only.

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$$\exists n \in \mathbb{N}\, \exists u \exists v\, (v_n u \in v^* \ \wedge\ \forall i \in \mathbb{Z}_+\ u_{n+i} \in uv^*)\,.$$

Thank You
very much!