Complexity of the hypercubic billiard

Nicolas Bedaride

Laboratoire d’Analyse Topologie Probabilités,

Université Paul Cézanne.

Complexity

If v is an infinite word, we define the **complexity** function $p(n, v)$ as the number of different words of length n inside v.

$$p : \mathbb{N}^* \longrightarrow \mathbb{N}$$

$$p : n \longmapsto p(n, v)$$

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Example : $u = abbbabaaa\ldots \qquad p(n, u) = 7 \quad \forall n \geq n_0$.

Sturmian word I

Theorem [Morse Hedlund 1940.] Let v be an infinite word, assume there exists n such that $p(n, v) \leq n$. Then v is an ultimately periodic word.

A word v such that $p(n, v) = n + 1$ for all integer n, is called a Sturmian word.

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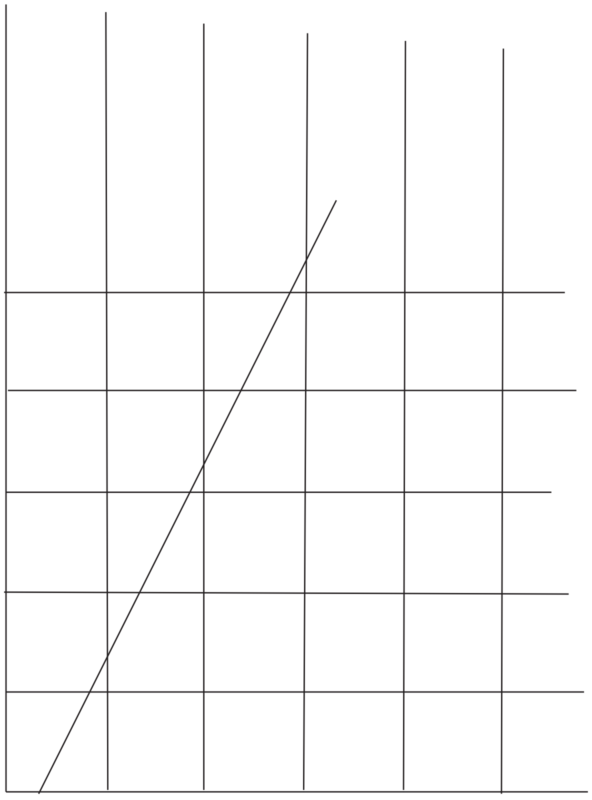
Theorem [Morse Hedlund 1940] We code a square with two letters. Let v be a sturmian word, then there exists m, ω in $\mathbb{R}^2$ such that $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{R}^2, \quad \frac{\omega_2}{\omega_1} \notin \mathbb{Q}$,

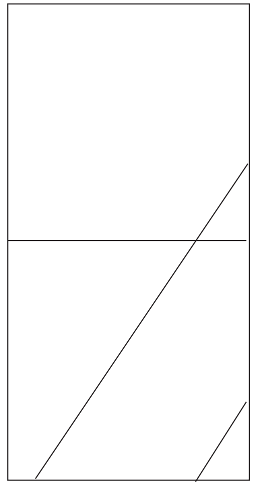
$$\phi(m, \omega) = v.$$

Sturmian word II



$$v = aabaabaab\ldots$$

Rotations



Sturmian word.

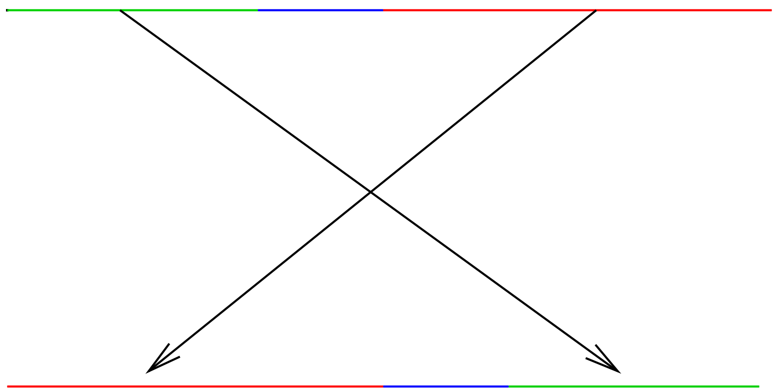
$$\Updownarrow$$

Rotation on the torus $\mathbb{T}^1$.

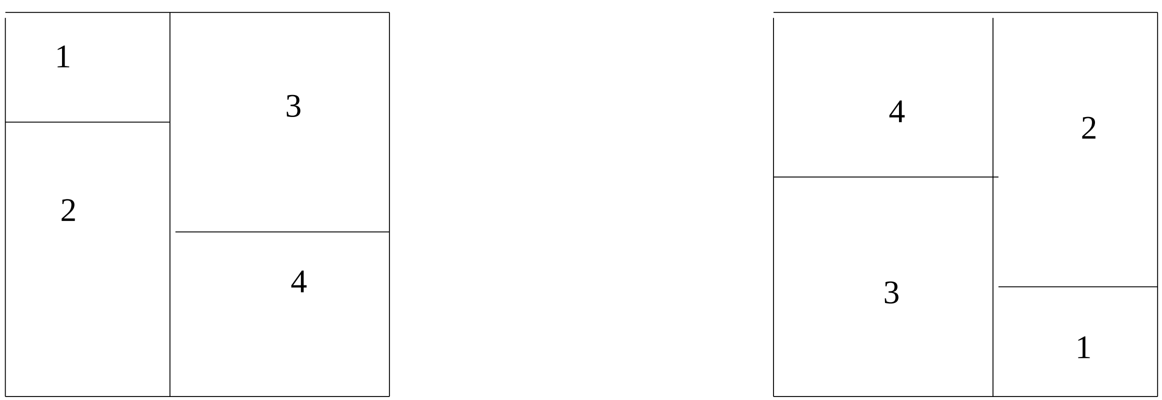
$$\Updownarrow$$

Two interval exchange.

Piecewise isometries



Interval exchange



Polygon exchange

Coding

Fix a point m, consider its orbit $(T^n(m))_{n\in\mathbb{N}}$. It is coded by a word v.

Assume T is a minimal map.

Computation of $p(n, v)$?

Entropy

Theorem [Buzzi 2002] If T is a piecewise isometry on $\mathbb{R}^d$ then

$$h_{top}(T) = \lim \frac{\log p(n)}{n} = 0.$$

Rotations

Two interval exchange : Rotation on the torus $\mathbb{T}^1$.

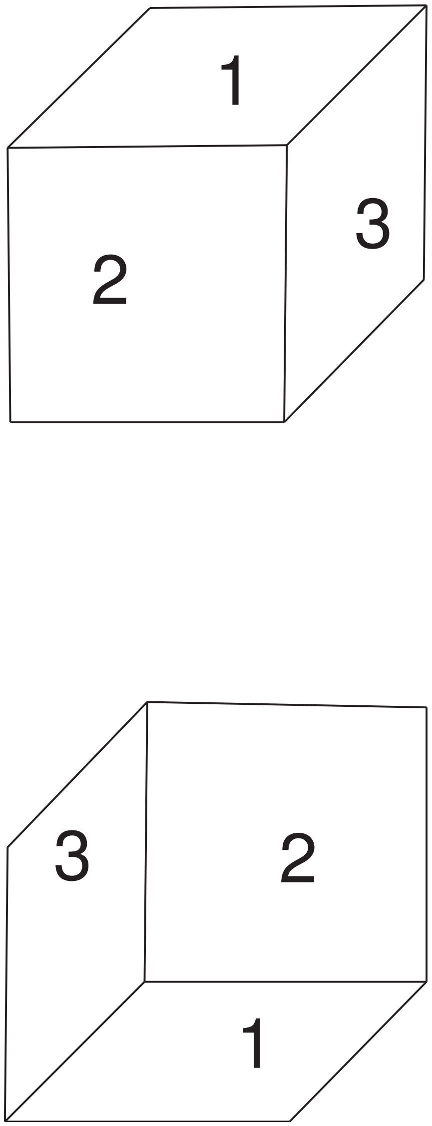
Three polygon exchange : Rotation on the torus $\mathbb{T}^2$.

Two interval exchange $\quad p(n, v) = n + 1.$

Three polygon exchange $\quad p(n, v) = n^2 + n + 1.$

Dimension d $\quad p(n, v) =?$

Polygons exchange



Notations

Rotation on the torus :

$$x \mapsto x + \omega[1],$$

$$\omega = (\omega_i)_{i \leq d}; x = (x_i)_{i \leq d}.$$

$$p(n, v) = p(n, \omega).$$

Results

If $d=2$ then $p(n,\omega)=n+1$.
If $d=3$ then :
Rauzy conjecture in 1980.
Arnoux, Mauduit, Shiokawa, Tamura in 1994.
Theorem [B2003] Assume the cube of $\mathbb{R}^3$ is coded by three letters. Assume ω fulfills following hypothesis :

$$(\omega_i)_{i\leq 3} \quad \text{independants over} \quad \mathbb{Q},$$

$$(\omega_i^{-1})_{i\leq 3} \quad \text{independants over} \quad \mathbb{Q},$$

Then

$$p(n,\omega)=n^2+n+1.$$

Result

Theorem [B2006] The cube of $\mathbb{R}^{d+1}$ is coded by $d+1$ letters. Assume ω fulfills following hypothesis :

$$(\omega_i)_{i \leq d+1} \quad \text{independants over} \quad \mathbb{Q},$$

$$(\omega_i^{-1})_{i \in I} \quad \text{independants over} \quad \mathbb{Q}\ \forall |I| = 3,$$

Then

$$p(n, d, \omega) = \sum_{i=0}^{min(n,d)} \frac{n!d!}{(n-i)!(d-i)!i!}.$$

Old and news proofs

- **Proof of baryshnikov in 1996** with the following hypothesis :

$$(\omega_i)_{i \le d+1} \quad \text{independants over} \quad \mathbb{Q},$$

$$(\omega_i^{-1})_{i \le d+1} \quad \text{independants over} \quad \mathbb{Q}.$$

We prove for $d \geq 2$:

$$s(n+1,d) - s(n,d) = d(d-1)p(n,d-2).$$

Complexity

Global method.

Lett $\mathcal{L}(n)$ a language, $p(n)$ its complexity function and $s(n) = p(n+1) - p(n)$. For $v \in \mathcal{L}(n)$ we introduce

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$$m_l(v) = card\{a \in \Sigma, \quad av \in \mathcal{L}(n+1)\}.$$

$$m_r(v) = card\{b \in \Sigma, \quad vb \in \mathcal{L}(n+1)\}.$$

$$m_b(v) = card\{a, b \in \Sigma, \quad avb \in \mathcal{L}(n+2)\}.$$

Definition A word v is called :
right special if $m_r(v) \geq 2$,
left special if $m_l(v) \geq 2$,
bispecial if it is right and left special.

We have

$$s(n) = \sum_{v \in \mathcal{L}(n)} (m_r(v) - 1).$$

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Cassaigne 97 Consider a factorial extendable language, then for all integer $n \geq 1$

$$s(n+1) - s(n) = \sum_{v \in \mathcal{BL}(n)} (m_b(v) - m_r(v) - m_l(v) + 1).$$

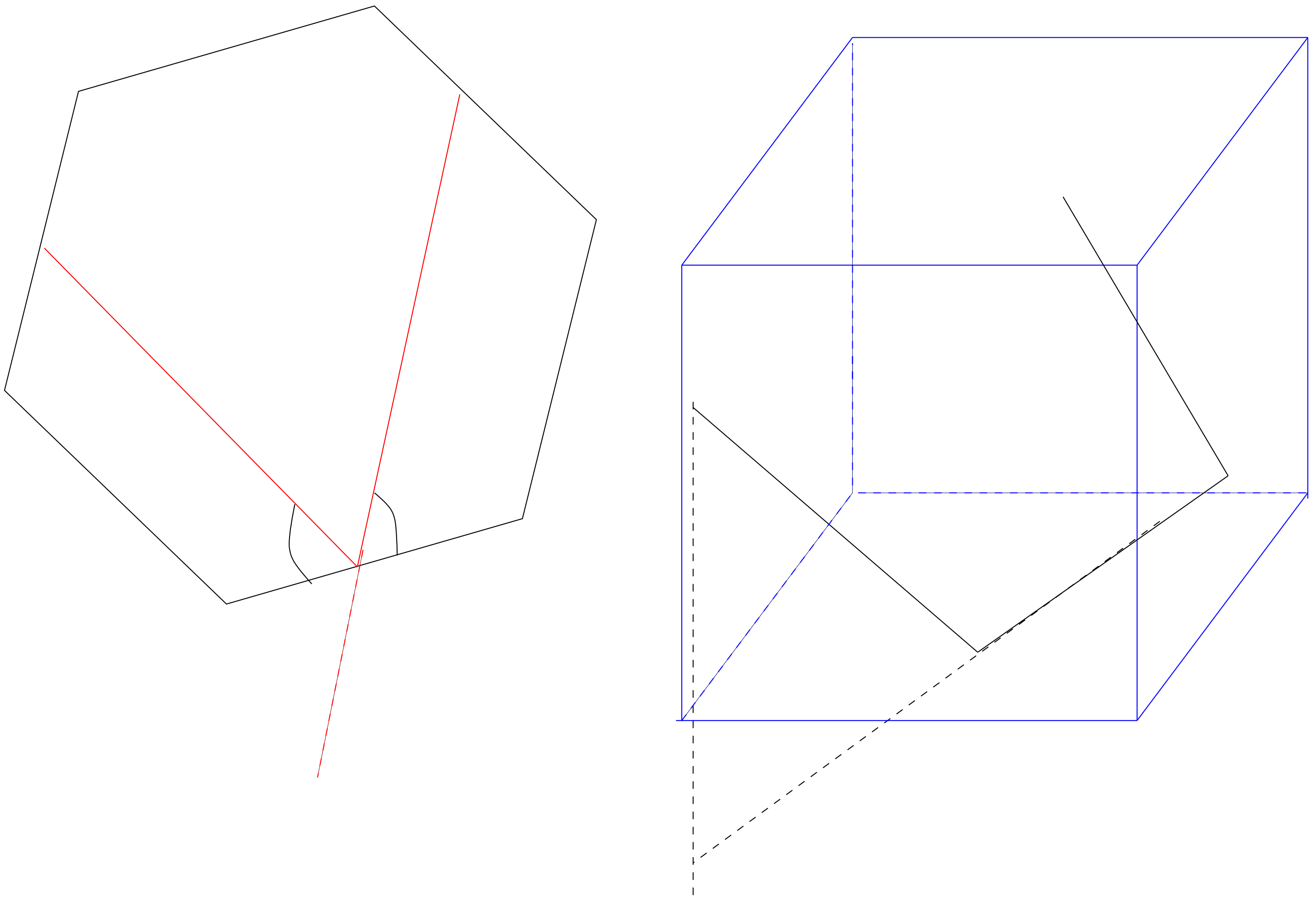
Billiard map

Let P be a polyhedron, $m \in \partial P$ and $\omega \in \mathbb{PR}^d$.
The point moves along a straight line until it reaches the boundary of P.
On the face : orthogonal reflection of the line over the plane of the face.

$$T : \quad X \longrightarrow \partial P \times \mathbb{PR}^d.$$

If a trajectory hits an edge, it stops.

Trajectories



Reflections and billiard.

Combinatorics

We label the faces of P by symbols from a finite alphabet. The symbols are called **letters**. The letters are elements of an **alphabet** Σ. After coding, the orbit of a point becomes an infinite **word**.

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Example : The periodic trajectory inside the square is coded by $acacac\ldots$.

$$p(n, v) = p(n, m, \omega) = p(n, \omega).$$

First return map

Consider the billiard map inside the cube of $\mathbb{R}^d$.

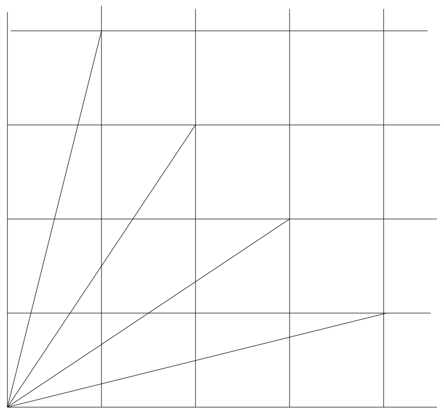
Identify the parallel faces.

Then the first return map to a transversal set is a rotation on the torus $\mathbb{T}^d$.

$$p(n, v) = p(n, \omega).$$

Diagonals

Definition Consider a polyhedron of $\mathbb{R}^3$. A diagonal between two edges A, B is the union of all billiard trajectories between A and B.
We say it is of length n if it intersects n faces between the two edges.



Diagonals of the square.

Case $d = 2$

Let A, B two edges of the cube. We can define diagonal in direction ω :

(0)

$$\gamma_{A,B,\omega} = \{a \in A, b \in B, (ab) \text{ is a billiard trajectory of length } n, \\ ab \text{ colinear } \omega\}.$$

We have

$$s(n+1, 2, \omega) - s(n, 2, \omega) = \sum_{\gamma(\omega)} \sum_{v \in \gamma} i(v).$$

$$s(n+1, 2, \omega) - s(n, 2, \omega) = 2.$$

Proof

A diagonal can contain several words if $d > 2$. We prove

$$s(n+1,d) - s(n,d) = \sum_{\gamma \in Diag} \sum_{v \in \gamma} i(v).$$

Geometry of $\gamma_{A,B,\omega}$.

If $d = 3$ then $dimA = dimB = 2$ and $dim\gamma_{A,B,\omega} = 2$.

Projection

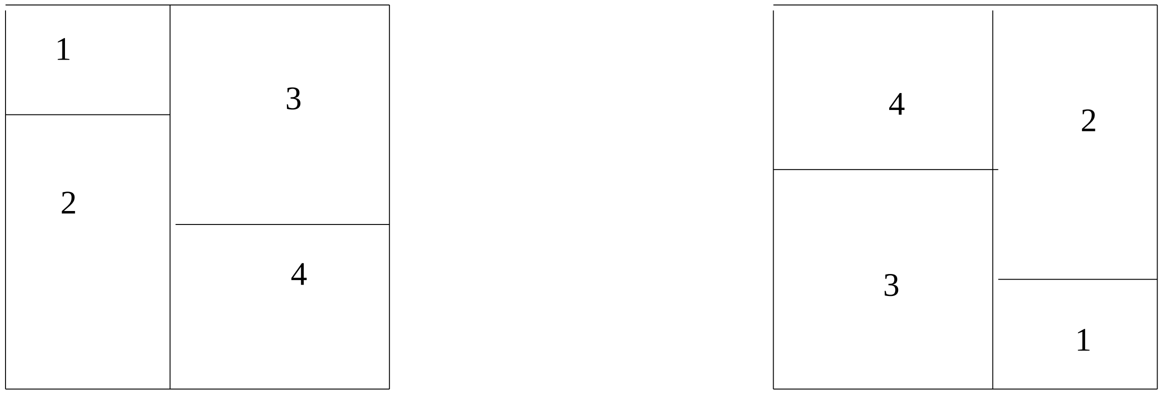
We use projection :
The orhtogonal projection of a billiard trajectory inside the cube is a billiard trajectory.
Projection of $\gamma_{A,B,\omega}$: billiard trajectory inside a cube of dimension $d-1$.

$$s(n+1,d,\omega) - s(n,d,\omega) = d(d-1)p(n,d-2,\omega').$$

Induction on the dimension.

Open questions

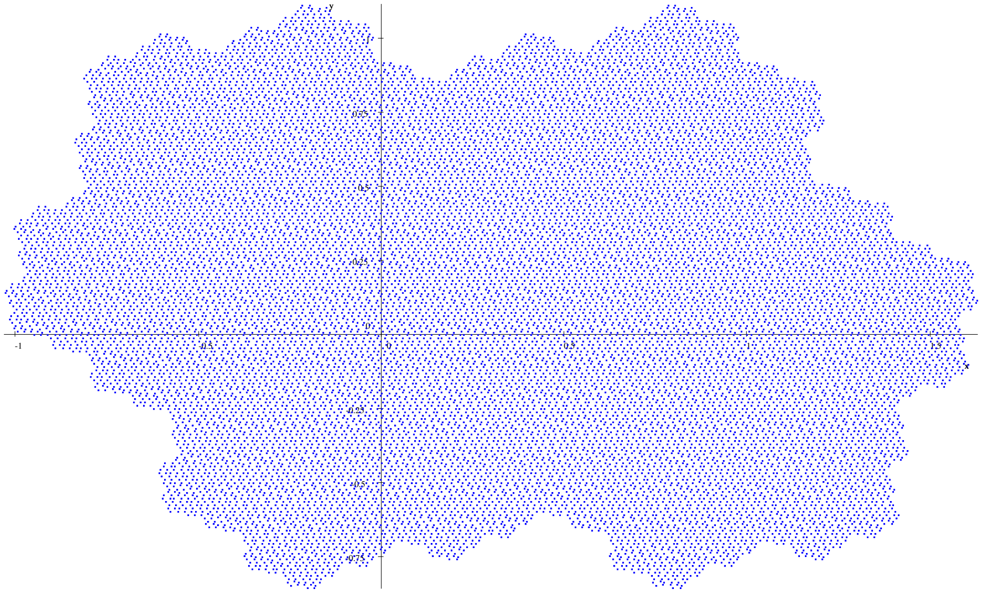
- Complexity of a rectangle exchange ?



- Combinatoric properties of rotation words in dimension $d \geq 3$.

- Piecewise isometries, dual billiard.

Rauzy fractal



$$p(n) = 2n + 1.$$