A Hierarchy of Automatic ω-Words having a decidable MSO Theory

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In this talk: finite descriptions using automata.

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over some alphabet Γ, such that $(D, \prec, \{P_a\}_{a \in \Sigma}) \cong W_w$.

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General facts

- The $\mathrm{FO}^{\mathrm{mod}}$ theory of every automatic structure is decidable.
- The class of automatic structures is closed under $\mathrm{FO}^{\mathrm{mod}}$-interpretations.

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In (generalized) numeration systems the usual (greedy) choice for $\prec$ is the length-lexicographic ordering

$$x <_{\text{llex}} y \quad \Longleftrightarrow \quad |x| < |y| \text{ or } |x| = |y| \text{ and } x <_{\text{lex}} y$$

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Proposition (Rigo,Maes '02)

An ω-word is morphic iff it is automatically presentable using $<_{llex}$.

Morphic words

An ω-word $w \in \Sigma^\omega$ is morphic if there is a morphism $\tau : \Gamma^* \to \Gamma^*$ with $\tau(a) = au$ for some $a \in \Gamma$ and a morphism $h : \Gamma^* \to \Sigma^*$ such that

$$w = h(\tau^\omega(a)) = h(a \cdot u \cdot \tau(u) \cdot \tau^2(u) \cdot \ldots \cdot \tau^n(u) \cdot \ldots) \quad .$$

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Examples

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- The fixed point of $\phi : a \mapsto ab, b \mapsto a$ is the *Fibonacci word* $f = a \cdot b \cdot a \cdot ab \cdot aba \cdot abaab \cdot abaababa \cdot \ldots$
- Consider $\tau : a \mapsto ab, b \mapsto ccb, c \mapsto c$ and $h : a, b \mapsto 1, c \mapsto 0$. Then

$$\tau^\omega(a) = a \cdot b \cdot ccb \cdot ccccb \cdot c^6 b \cdot \ldots$$

and $h(\tau^\omega(a))$ is the characteristic sequence of the set of squares.

Theorem (cf. Rabinovich, Thomas '06)

The MSO theory of W_w is decidable iff there is a recursive factorization

$$w = \underset{f(0)}{} \; w_0 \underset{f(1)}{\cdot} w_1 \underset{f(2)}{\cdot} \ldots \underset{f(n)}{\cdot} w_n \underset{f(n+1)}{\cdot} \underset{\ldots}{\ldots}$$

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such that for every morphism ψ into a finite monoid M the contraction of w wrt. ψ and f:

$$w_f^{\psi} = \psi(w_0) \cdot \psi(w_1) \cdot \ldots \cdot \psi(w_n) \cdot \ldots \in M^{\omega}$$

is ultimately periodic (with both period and threshold computable from ψ).

Deciding the MSO theory of morphic words [Carton,Thomas '02]

Consider

$$w = h(a \cdot u \cdot \tau(u) \cdot \tau^2(u) \cdot \ldots \cdot \tau^n(u) \cdot \ldots)$$

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$$w^{\psi}_{f_\tau} = \psi(h(a)) \cdot \psi(h(u)) \cdot \psi(h(\tau(u))) \cdot \psi(h(\tau^2(u))) \cdot \ldots \cdot \psi(h(\tau^n(u))) \cdot \ldots,$$

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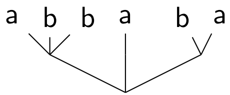
is ultimately periodic, since there are (computable) N and p such that

$$\psi \circ h \circ \tau^{n+p} = \psi \circ h \circ \tau^n \quad (n > N)$$

Morphisms of k stacks

k-stacks as parenthesized words — or — as trees of height k

$$[\,[abb]\,[a]\,[ba]\,]$$



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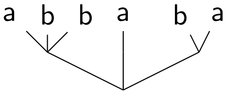

Morphisms of k-stacks $\approx$ k-stack of morphisms:

$$\mathrm{Stack}_\Gamma^{(0)} = \Gamma \qquad \mathrm{Hom}_\Gamma^{(0)} = \Gamma \to \Gamma$$
$$\mathrm{Stack}_\Gamma^{(k+1)} = [(\mathrm{Stack}_\Gamma^{(k)})^*] \qquad \mathrm{Hom}_\Gamma^{(k+1)} = [(\mathrm{Hom}_\Gamma^{(k)})^*]$$ (uniformity!)

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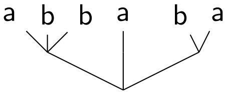

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Application:

- $\varphi^{(0)}(\gamma^{(0)})$ is as given,
- for $\varphi^{(k+1)} = [\varphi_1^{(k)} \ldots \varphi_s^{(k)}]$ and $\gamma^{(k+1)} = [\gamma_1^{(k)} \ldots \gamma_t^{(k)}]$

$$\varphi^{(k+1)}(\gamma^{(k+1)}) = [\varphi_1^{(k)}(\gamma_1^{(k)})...\varphi_s^{(k)}(\gamma_1^{(k)}) \cdots \varphi_1^{(k)}(\gamma_t^{(k)})...\varphi_s^{(k)}(\gamma_t^{(k)})]$$

k-Morphic words

An word $w \in \Sigma^\omega$ is k-morphic if there is a morphism $\varphi \in \mathrm{Hom}_\Gamma^{(k)}$, a k-stack $\gamma \in \mathrm{Stack}_\Gamma^{(k)}$, and a homomorphism $h : \Gamma^* \to \Sigma^*$ such that

$$w = h\left(\prod_{n=0}^{\infty} \mathrm{leaves}(\varphi^n(\gamma))\right) \quad .$$

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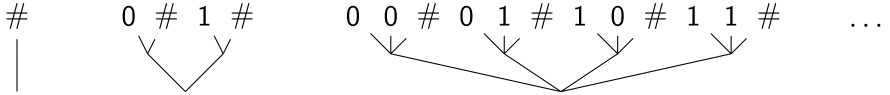
Example

Let $\gamma = [[\#]]$, $\varphi = [\varphi_0\varphi_1]$ *with* $\varphi_i : \left| \begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ \# & \mapsto & i\# \end{array} \right.$. *(Non-uniform!)*



Similarly, $s = 12345678910111213\ldots$ (Champernowne word) is 2-morphic.

k-length-lexicographic presentations

Consider $u = a_0 a_1 \ldots a_{tk-1} \in \Sigma^{tk}$.
Its k-split is $(u^{(1)}, \ldots, u^{(k)})$ with $u^{(i+1)} = a_i a_{k+i} \ldots a_{(t-1)k+i}$ f.a. $i < k$.
Additionally, let $u^{(0)} = 1^{|u|}$.
Conversely, $u = \otimes_k(u^{(1)}, \ldots, u^{(k)})$ is the k-shuffle of the $u^{(i)}$-s.

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For $0 \leq i < k$ we define the equivalence

$$u =_i v \stackrel{\text{def}}{\Longleftrightarrow} \forall j \leq i \quad u^{(j)} = v^{(j)} \quad \text{(implying } |u| = |v|\text{)}.$$

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Consider some lin. ord. $<$ of Σ with induced $<_{\text{lex}}$.
The induced k-length-lexicographic ordering $<_{k\text{-llex}}$ is defined as

$$u <_{k\text{-llex}} v \overset{\text{def}}{\Longleftrightarrow} |u| < |v| \vee \exists i < k : \; u =_i v \wedge u^{(i+1)} <_{\text{lex}} v^{(i+1)} \; .$$

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Theorem

For all k, an ω-word is k-morphic iff it has an aut. pres. using $<_{k\text{-}llex}$.

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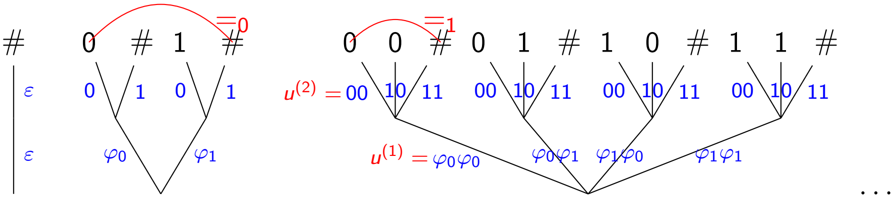

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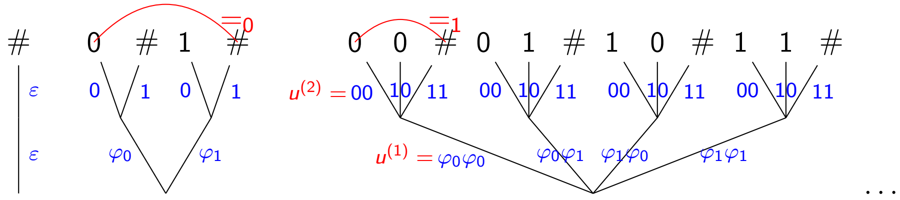

Notation

For each k, $\mathcal{W}_k$ is the class of k-morphic, or k-lex, words.

Hierarchy theorem

Clearly, $\mathcal{W}_k \subseteq \mathcal{W}_{k+1}$.

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s_0 &= a^\omega \\
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s_2 &= abcaabaabc(a^4b)^4c(a^8b)^8c\ldots \\
s_3 &= abcd((a^2b)^2c)^2d((a^4b)^4c)^4d((a^8b)^8c)^8d\ldots \\
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Theorem (Hierarchy Theorem)

For each $k \in \mathbb{N}$ we have $s_{k+1} \in \mathcal{W}_{k+1} \setminus \mathcal{W}_k$.

Deciding the MSO theory of k-morphic words

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Proof plan

$(=_0, ..., =_k)$ provide a "built in" factorization of depth k of each $w \in \mathcal{W}_{k+1}$

Contraction Lemma

For all $w \in \mathcal{W}_{k+1}$ and ψ we have $w^{\psi}_{=_k} \in \mathcal{W}_k$ effectively.

By iterated contractions $w^{\psi}_{=_i} \in \mathcal{W}_i$, in particular, $w^{\psi}_{=_0}$ is ultimately periodic.

Contraction Lemma

Illustration for $k = 1$:

w generated by $\varphi = [\varphi_0\varphi_1]$ with $\varphi_i \in \mathrm{Hom}(\Sigma^\star, \Sigma^\star)$ and $\gamma = [[a]]$:

$$w = a \cdot a\varphi_0\, a\varphi_1 \cdot a\varphi_0\varphi_0\, a\varphi_0\varphi_1\, a\varphi_1\varphi_0\, a\varphi_1\varphi_1 \cdot a\varphi_0\varphi_0\varphi_0\, a\varphi_0\varphi_0\varphi_1 \ldots$$

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Higher-Order Regularity Lemma

Consider Θ a finite set of morphisms on $\Sigma^\star$, $\vartheta : \Theta^\star \to \mathrm{Hom}(\Sigma^\star, \Sigma^\star)$ s.t. $\vartheta(x \cdot y) = \vartheta(y) \circ \vartheta(x)$. Let $\psi \in \mathrm{Hom}(\Sigma^\star, M)$, $a \in \Sigma$, $m \in M$.

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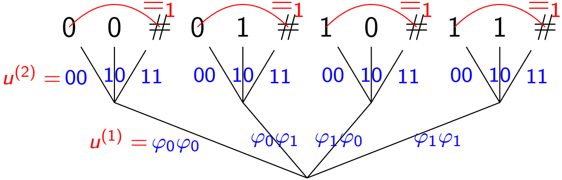
$\varphi_i : \left| \begin{array}{lll} 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ \# & \mapsto & i\# \end{array} \right.$
$\psi(x) = |x|_1 \bmod 2$



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$\gamma = [0]$

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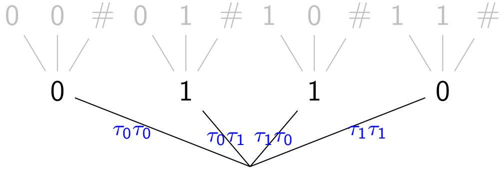

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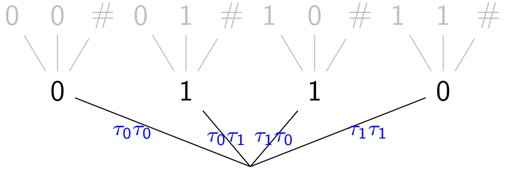
Higher-Order Regularity Lemma

Consider Θ a finite set of morphisms on $\Sigma^\star$, $\vartheta : \Theta^\star \to \mathrm{Hom}(\Sigma^\star, \Sigma^\star)$ s.t. $\vartheta(x \cdot y) = \vartheta(y) \circ \vartheta(x)$. Let $\psi \in \mathrm{Hom}(\Sigma^\star, M)$, $a \in \Sigma$, $m \in M$.
Then $L_{\vartheta,\psi,a,m} = \{x \in \Theta^\star \mid \psi(\vartheta(x)(a)) = m\}$ is regular. (effective)

$\gamma = [0]$

τ :

		τ_0	τ_1
0	$\mapsto$	0	1
1	$\mapsto$	1	0



For $k = 0$: $\Theta = \{\tau\}$ is unary and the $L_{\vartheta,\psi,a,m}$ are ultimately periodic.

Main Results

Theorem (Main Theorem)

Given a k-lex. presentation of $w \in \mathcal{W}_k$ and $\varphi(\vec{x}) \in \mathrm{MSO}$ having only first-order variables $\vec{x}$ free, we can compute an automaton recognizing the relation defined by φ in W_w.

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Corollaries

- Each $\mathcal{W}_k$ is closed under MSO-definable recolorings.
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For each k consider $w_k \in \{0, 1, \#\}^\omega$ obtained by concatenating all finite binary words in the k-lexicographic ordering and separated by hash marks.

Theorem (Characterization)

Let $w \in \Sigma^\omega$. Then $w \in \mathcal{W}_k \iff W_w \leq^{\mathcal{I}} W_{w_k}$ for some interpretation $\mathcal{I} = (\varphi_D(x), x < y, \{\varphi_a(x)\}_{a \in \Sigma})$ such that $\models \forall x(\varphi_D(x) \to P_{\#}(x))$.

Future work and Questions

To do

- Locate $\mathcal{W}_k$ in the pushdown hierarchy...
  or generate them from simply-typed schemes.
- Extend results to other (all?) automatic presentations of $(\mathbb{N}, <)$...
  to other linear orderings....

$**$ Is isomorphism of k-lexicographic words decidable?

$**$ Let $k > k'$. Is it decidable whether a k-morphic word is k'-lexicographic?
In particular, is eventual periodicity of k-morphic words decidable?

(Cf. same problems for ω-words generated by HD0L systems, i.e. $k = 1$)

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THANK YOU!