A new algebraic invariant for weak equivalence of sofic subshifts*

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Abstract

It is studied how taking the inverse image by a sliding block code affects the syntactic semigroup of a sofic subshift. Two independent approaches are used: ζ-semigroups as recognition structures for sofic subshifts, and relatively free profinite semigroups. A new algebraic invariant is obtained for weak equivalence between sofic subshifts, by determining which classes of sofic subshifts naturally defined by pseudovarieties of finite semigroups are closed under weak equivalence. Among such classes are the classes of almost finite type subshifts and aperiodic subshifts. The algebraic invariant is compared with other robust conjugacy invariants.

1 Introduction

Dynamical systems were first introduced in order to study systems of differential equations used to model physical phenomena. When discretizing both time and space, the physical system becomes a "symbolic" dynamical system that yields information on the real one. Symbolics dynamics is a rapidly growing area that borrows its methods from various fields such as combinatorics, algebra, automata theory, probabilities, etc, and finds some far-reaching applications in coding theory, data storage and transmission, linear algebra...

The systems of symbolics dynamics or *subshifts*, are sets of bi-infinite words, topologically closed and invariant under a *shift* operation. When trying to classify these systems, there happens to be a natural notion of equivalence between them, called *conjugacy*. Unfortunately, despite a rich and prestigious litterature on the subject, the decidability of conjugacy remains wide open, even for subclasses of simple systems such as systems of finite type or sofic systems. To try to cope with this major difficulty, some weaker notions of equivalence between subshifts were introduced: see [18, 6]. The *shift equivalence* has been the most important of them; it is decidable, although the algorithm is quite intricate. In this paper, we focus on the *weak equivalence*, defined by Béal and Perrin

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in [6], which relies on inverse images of sliding block codes (the morphisms between subshifts). We deduce an algebraic invariant for weak equivalence for sofic subshifts. Moreover, we exhibit a pair of two sofic subshifts for which that invariant is used to easily prove that they are not weak equivalent, while various robust conjugacy invariants fail to detect that they are not conjugate. The significance of this example is more appreciated once we realize that the weak equivalence relation really deserves its name, in the sense that there are very general sufficient conditions for two subshifts to be weak equivalent [6].

We briefly sketch the nature of our algebraic invariant. There is a natural bijective correspondence between subshifts and factorial prolongable languages. The sofic subshifts are precisely those subshifts whose corresponding language is rational. A well established method of classification of rational languages is by grouping them in *varieties of rational languages*. By the well known Eilenberg's Correspondence Theorem, varieties of languages are in a natural correspondence with pseudovarieties of finite semigroups. In this way a pseudovariety of finite semigroups defines naturally a class of sofic subshifts. In [13] it was determined which of these classes are closed under taking conjugate subshifts. It was also proved that such classes are closed under shift equivalence. In this paper we prove they are also closed under weak equivalence. The arguments used in [13] are based in the equational description of a pseudovariety using pseudoidentities. These arguments are somewhat heavy and it seems difficult to adapt them for the weak equivalence case, hence in this paper we use different approaches.

The paper is organized as follows. Preliminary definitions and results are made in Section 2. Division between subshifts and weak equivalence are introduced in Section 3. Section 4 deals with the perspective of the first author Master's Thesis [11] of seeing finite ζ-semigroups (a generalization of ω-semigroups) as recognition structures for sofic subshifts. ¿From a Theorem of [11] about how the operation of taking the inverse image of a subshift by a sliding block is reflected in the corresponding syntactic ζ-semigroups we deduce a similar result concerning syntactic semigroups in the usual sense. In Section 5 we obtain our algebraic invariant, and using it we list some important classes of sofic subshifts closed under taking weak equivalent subshifts. The dynamic significance of this algebraic invariant is evaluated in Section 6. The content from Section 5 is recovered in Section 7 with results about relatively free profinite semigroups, with great proof economy. This approach complements the one using ζ-semigroups, which demanded a longer and heavier preparation, but produced more intermediate results, of a more precise nature. Another advantage of ζ-semigroups is that with a little additional effort one can use this approach to generalize results about semigroups to results about *ordered semigroups* (see [21] and [22] for details about ordered semigroups). However, we shall focus only in the unordered case, since this one is rich enough.

As general references for symbolic dynamics see [3, 18]. For semigroup theory, rational languages and finite automata see [20, 1, 2].

2 Preliminaries

2.1 Subshifts and codes

Let A be an alphabet. All alphabets in this paper are assumed to be finite. Let $A^{\mathbb{Z}}$ be the set of sequences of letters of A indexed by $\mathbb{Z}$. A factor of an element $(x_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ is a finite sequence $x_k x_{k+1}\cdots x_{k+n-1}x_{k+n}$, denoted by $x_{[k,k+n]}$, where $k\in\mathbb{Z}$ and $n\geq 0$. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology of A. Recall that the topology of $A^{\mathbb{Z}}$ is characterized by the fact that a sequence $(x^{(n)})_n$ of elements of $A^{\mathbb{Z}}$ converges to x if and only if for every positive integer k there is p_k such that $n\geq p_k$ implies $(x^{(n)})_{[-k,k]}=x_{[-k,k]}$. Note that $A^{\mathbb{Z}}$ is a compact Hausdorff space. From here on, *compact* will mean both compact and Hausdorff.

The *shift* in $A^{\mathbb{Z}}$ is the bijective function σ_A (or just σ) from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ defined by $\sigma_A((x_i)_{i\in\mathbb{Z}})=(x_{i+1})_{i\in\mathbb{Z}}$. A *shift dynamical system* or *subshift* of $A^{\mathbb{Z}}$ is a closed subset $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that $\sigma_A(\mathcal{X})\subseteq\mathcal{X}$ and $\sigma_A^{-1}(\mathcal{X})\subseteq\mathcal{X}$.

Denote by $A^{\tilde{\omega}}$ (respectively A^{ω}) the set of sequences of letters of A indexed by the set of negative integers (respectively non-negative integers). The map $\varphi: x\mapsto(\ldots x_{-3}x_{-2}x_{-1},x_0x_1x_2)$ is an homeomorphism between $A^{\mathbb{Z}}$ and the product space $A^{\tilde{\omega}}\times A^{\omega}$. The sequence $\varphi^{-1}(z,t)$ is usually denoted by $z.t$.

If $\mathcal{X}$ is a subset of $A^{\mathbb{Z}}$ then we denote by $L(\mathcal{X})$ the set of finite factors of elements of $\mathcal{X}$. There is a subset $\mathcal{F}$ of A^+ such that $L(\mathcal{X})=A^+\setminus A^*\mathcal{F}A^*$; a set $\mathcal{F}$ in such conditions is called a set of *forbidden words* for $\mathcal{X}$. A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is of *finite type* if it has a finite set of forbidden words. An element x of $A^{\mathbb{Z}}$ belongs to $\mathcal{X}$ if and only if every finite factor of x belongs to $L(\mathcal{X})$. The correspondence $\mathcal{X}\mapsto L(\mathcal{X})$ is a bijection between the set of subshifts of $A^{\mathbb{Z}}$ and the set of factorial prolongable languages of A^+.

A *sliding block code* (or more briefly, a *code*) F between the subshifts $\mathcal{X}$ of $A^{\mathbb{Z}}$ and $\mathcal{Y}$ of $B^{\mathbb{Z}}$ is a function $F:\mathcal{X}\to\mathcal{Y}$ for which there are integers $k,l\geq 0$ and a function $f:A^{k+l+1}\to B$ such that $F(x)=(f(x_{[i-k,i+l]}))_{i\in\mathbb{Z}}$. If we can choose f such that $k+l+1=n$ then we say that F has window size n. We say that f is a *block map* of F with *memory* k and *anticipation* l. The sliding block code F depends only on the restriction of f to $A^{k+l+1}\cap L(\mathcal{X})$.

It is well known [16] that a map $F:\mathcal{X}\subseteq A^{\mathbb{Z}}\to\mathcal{Y}\subseteq B^{\mathbb{Z}}$ between subshifts is a code if and only if it is a continuous function such that $F\circ\sigma_A=\sigma_B\circ F$. Note that the identity transformation of a subshift is a code, the composition of two codes is a code and the inverse of a bijective code is a code. A bijective code is called a *conjugacy*. Two subshifts are *conjugate* if there is a conjugacy between them. A *conjugacy invariant* is a property of subshifts that is preserved for taking conjugate subshifts. See [18] for the definition and computation of ordinary conjugacy invariants like the zeta function.

Two codes $\varphi_1:\mathcal{X}_1\to\mathcal{Y}_1$ and $\varphi_2:\mathcal{X}_2\to\mathcal{Y}_2$ are said to be *conjugate* if there are conjugacies $f:\mathcal{X}_1\to\mathcal{X}_2$ and $g:\mathcal{Y}_1\to\mathcal{Y}_2$ such that $\varphi_2\circ f=g\circ\varphi_1$. We say that the pair (f,g) is a *conjugacy* between φ_1 and φ_2.

Given an alphabet A and $k\geq 1$, consider the alphabet A^k. To avoid ambiguities, we represent an element $w_1\ldots w_n$ of $(A^k)^+$ (with $w_i\in A^k$) by $\langle w_1,\ldots,w_n\rangle$. For $k\geq 0$ let Φ_k be the function from A^+ to $(A^{k+1})^*$ defined by

$$\Phi_k(a_1\ldots a_n)=\begin{cases}1 & \text{if } n\leq k,\\ \langle a_{[1,k+1]},a_{[2,k+2]},\ldots a_{[n-k-1,n-1]},a_{[n-k,n]}\rangle & \text{if } n>k,\end{cases}$$

where $a_i \in A$ and $a_{[i,j]} = a_i a_{i+1} \dots a_{j-1} a_j$. For a map $f : A^k \to B$, let $\hat{f}$ be the unique monoid homomorphism between $(A^k)^*$ and B^* extending f. Let $\bar{f}$ be $\hat{f} \circ \Phi_{k-1}$. Our interest in $\bar{f}$ relies on the fact that if $F : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ is a code with memory k and anticipation l then $F(x)_{[i,j]} = \bar{f}(x_{[i-k,i+l]})$.

By *graph* we mean an oriented graph. A *labeled graph* (G, π) is a pair such that G is a graph and π is a function mapping edges of G into letters of an alphabet A. We consider (G, π) as an automaton such that all states are initial and final, recognizing the words that are labels of paths of G through the map π. We say that a labeled graph *presents* the subshift $\mathcal{X}$ if it recognizes $L(\mathcal{X})$. A (labeled) graph is *essential* if all vertices lie in a bi-infinite path on the graph. A subshift $\mathcal{X}$ is *sofic* if $L(\mathcal{X})$ is rational. Note that finite type subshifts are sofic. One can see that $\mathcal{X}$ is sofic if and only if $L(\mathcal{X})$ is recognized by an essential finite labeled graph. For a finite graph G, let E be the set of its edges. The subset X_G of $E^{\mathbb{Z}}$ whose finite factors are paths of G is a finite type subshift of $E^{\mathbb{Z}}$. Given a subshift $\mathcal{Y}$ presented by a labeled graph (G, π), let π_* the map from X_G to $\mathcal{Y}$ that maps a sequence $(e_i)_{i \in \mathbb{Z}}$ into $(\pi(e_i))_{i \in \mathbb{Z}}$. Then π_* is an onto code with window size zero. We call π_* the *cover* associated with (G, π).

A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if for all $u, v \in L(\mathcal{X})$ there is $w \in A^*$ such that $uwv \in L(\mathcal{X})$. Irreducibility is a conjugacy invariant. A sofic subshift is irreducible if and only if it is presented by a strongly connected finite labeled graph [15]. We consider now a stronger property. A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *mixing* if for all $u, v \in L(\mathcal{X})$ there is an integer N such that for all $n \geq N$ there is $w \in A^*$ with length n such that $uwv \in L(\mathcal{X})$. Being irreducible or mixing is a property preserved for taking images under codes.

The *Krieger cover* of a sofic subshift $\mathcal{X}$ is the cover associated with the essential labeled graph obtained from the minimal deterministic automaton of $L(\mathcal{X})$ by deleting states that do not lie in bi-infinite paths. Krieger proved in [17] that two sofic subshifts are conjugate if and only if their Krieger covers are conjugate. If the sofic subshift $\mathcal{X}$ is irreducible then the labeled graph representing its Krieger cover has a unique terminal strongly connected component which is an essential labeled graph presenting $\mathcal{X}$ [7]. The corresponding cover is the *Fischer cover* of $\mathcal{X}$. Two irreducible sofic subshifts are conjugate if and only if their Fischer covers are conjugate.

2.2 Semigroups

Recall that an element e of a semigroup S is *idempotent* if $e^2 = e$. If S is finite then for every $s \in S$ the set of the powers s^n (with n positive integer) has a unique idempotent of S.

A semigroup S *divides* a semigroup T if S is a homomorphic image of subsemigroup of T. We also say that S is a *divisor* of T. This situation is denoted by $S \prec T$. A *pseudovariety of semigroups* is a class of finite semigroups closed under taking divisors and finite direct products.

Example 2.1. *The following classes are pseudovarieties of semigroups:*

(1) the class Com *of finite commutative semigroups;*
(2) the class Sl *of finite commutative idempotent semigroups;*
(3) the class N *of finite* nilpotent *semigroups, that is, finite semigroups with a zero as sole idempotent;*

(4) *the class* Inv *of finite semigroups whose idempotents commute;*
(5) *the class* A *of finite* aperiodic *semigroups, that is, finite semigroups whose subgroups are trivial;*
(6) *the class* D_k *of finite semigroups satisfying the identity* $xy_1 \cdots y_k = y_1 \cdots y_k$;
(7) *the class* D *of finite semigroups whose idempotents are right zeros; one has* $\mathsf{D} = \bigcup_{k \geq 1} \mathsf{D}_k$;
(8) *for every pseudovariety* V *of semigroups, the class* LV *of semigroups whose subsemigroups that are monoids belong to* V.

Let L be a language of A^+. The *context* of a word u of A^+ relatively to L is the set $C_L(u) = \{(x, y) \in A^* \times A^* \mid xuy \in L\}$. The *syntactic semigroup* of L is the quotient of A^+ by the congruence $\equiv_L$ defined by $u \equiv_L v \Leftrightarrow C_L(u) = C_L(v)$.

A language L of A^+ is *recognized* by a semigroup homomorphism $\varphi : A^+ \to S$ if there exists a subset I of S such that $L = \varphi^{-1}(I)$. We say that L is recognized by S if there is a semigroup homomorphism $\varphi : A^+ \to S$ recognizing L. The syntactic semigroup of L recognizes L and divides all semigroups recognizing L. A *recognizable* or *rational* language is a language recognized by a finite semigroup. It is well known that a language L is rational if and only if L is recognized by a finite automaton, if and only if its syntactic semigroup is finite.

Consider a pseudovariety of semigroups V. A V-*recognizable* language of A^+ is a language recognized by a semigroup from V. A language is V-recognizable if and only if its syntactic semigroup belongs to V.

Let S and T be semigroups. The set S^T of maps from T to S, viewed as a direct product of copies of S, is a semigroup; the product fg between two elements f and g of S^T is defined by the rule $fg(t) = f(t)g(t)$.

For a semigroup T, denote by T^1 the monoid that equals T if T is a monoid, and if not then $T^1 = T \cup \{1\}$ for some extra element 1, with the semigroup operation of T^1 extending that of T and 1 being the neutral element of T^1.

For this paragraph, see [1, Chapter 10] or [14]. Given semigroups S and T, let $t_0 \in T^1$ and $f \in S^{T^1}$. Denote by ${}^{t_0}f$ the element of S^{T^1} given by the correspondence $t \mapsto f(tt_0)$. The *wreath product* of S and T, denoted by $S \circ T$, is the semigroup with underlying set $S^{T^1} \times T$ and the following operation:

$$(f_1, t_1) \cdot (f_2, t_2) = (f_1 \cdot {}^{t_1}f_2, t_1 \cdot t_2).$$

The *semidirect product* of two pseudovarieties V and W, denoted by $\mathsf{V} * \mathsf{W}$, is the class of divisors of semigroups of the form $S \circ T$, with $S \in \mathsf{V}$ and $T \in \mathsf{W}$. The class $\mathsf{V} * \mathsf{W}$ is also a pseudovariety. The semidirect product of pseudovarieties is associative. One has $\mathsf{D} * \mathsf{D} \subseteq \mathsf{D}$, $\mathsf{V} * \mathsf{D} \subseteq \mathsf{LV}$ and $\mathsf{LV} = \mathsf{LV} * \mathsf{D}$. Also, $\mathsf{LSI} = \mathsf{SI} * \mathsf{D}$.

3 Weak equivalence

Confronted with the difficulty of deciding conjugacy, some other equivalence relations between subshifts were introduced such as the *weak equivalence* defined by M.-P. Béal and D. Perrin in [6].

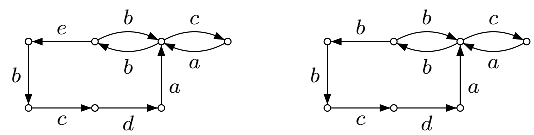
Let A, B be two alphabets, let \$ be a symbol that does not belong to B and let $B_\$ = B \cup \{\$\}$. A subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ will be said to *divide* a subshift $\mathcal{Y} \subseteq B^{\mathbb{Z}}$ - denoted $\mathcal{X} \prec \mathcal{Y}$ - if there exists a sliding block code $F : A^{\mathbb{Z}} \to {B_\$}^{\mathbb{Z}}$ such that $\mathcal{X} = F^{-1}(\mathcal{Y})$; we also say that $\mathcal{X}$ is a *divisor* of $\mathcal{Y}$. Two subshifts $\mathcal{X}$ and $\mathcal{Y}$ are *weak equivalent* if $\mathcal{X} \prec \mathcal{Y}$ and $\mathcal{Y} \prec \mathcal{X}$.

It was proved in [6] that both classes of sofic subshifts and of subshifts of finite type are closed under weak equivalence. The properties of beeing mixing or irreducible are not weak equivalence invariants [6]. The following result shows that the weak equivalence is indeed weaker than conjugacy of subshifts. Its proof, although unpublished, was put forward to us by M.-P. Béal.

Theorem 3.1. *Two conjugate subshifts are weak equivalent.*

It is essential to notice that the relation of division between subshifts cannot be reduced to a similar relation between the corresponding languages of finite factors. Let us be more precise. Let $\mathcal{X}$ and $\mathcal{Y}$ be subshifts of $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$, respectively. Write $\mathcal{X} \triangleleft \mathcal{Y}$ if there is an integer n and a map $f : A^n \to B_\$$ such that $L(\mathcal{X}) \setminus A^{<n} = \bar{f}^{-1}(L(\mathcal{Y}))$. Then we have the following result:

Proposition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be the following irreducible sofic subshifts:*



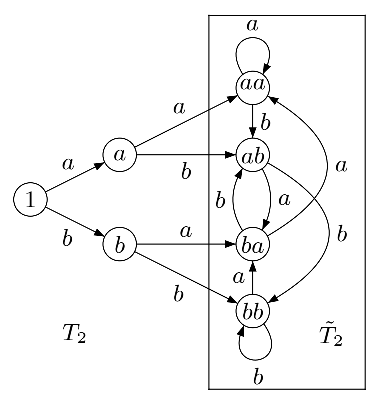
Then $\mathcal{X}$ and $\mathcal{Y}$ are conjugate but $\mathcal{X} \not\triangleleft \mathcal{Y}$.

4 ζ-semigroups

Given an alphabet A and a non-negative integer k, consider the *De Bruijn automaton* $T_k(A)$ whose states are the words of $A^{\leq k}$ and where each letter a of A acts deterministically over every state u as follows:

$$\begin{cases} u \cdot a = ua, \text{ if } |u| < k, \\ u \cdot a = v, \text{ if } |u| = k \text{ and } ua = bv \text{ for some } b \in A. \end{cases}$$

The transition semigroup of $T_k(A)$, denoted by $\mathcal{D}_k$, is in the pseudovariety D_k since for each finite word x of length k and each state q of T_k, $q \cdot x = x$. Define now the sub-automaton $\tilde{T}_k(A)$, built from $T_k(A)$ by deleting states corresponding to words of $A^{<k}$. The transition semigroup $\tilde{\mathcal{D}}_k$ of $\tilde{T}_k(A)$ is in D_k as well. The following figure illustrates both constructions for $k = 2$.



Consider a map $f : A^k \to B$. The semigroup $\mathcal{D}_{k-1}$ is relevant to characterize the inverse image of a rational language under $\bar{f}$. The following theorem is a particular instance of a classical result of semigroup theory [14].

Theorem 4.1. *For alphabets A and B and a positive integer k, consider a map $f : A^k \to B$. Let $Y \subseteq B^+$, $X \subseteq A^+$ be two rational languages. Denote by $S(X)$ (resp. $S(Y)$) the syntactic semigroup of X (resp. Y). Then $X = \bar{f}^{-1}(Y)$ implies $S(X) \prec S(Y) \circ \mathcal{D}_{k-1}$.*

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. For the sake of conciseness, the syntactic semigroup of $L(\mathcal{X})$ will be called the *syntactic semigroup of* $\mathcal{X}$. When we consider the inverse image by a sliding block code of a sofic subshift do we have a result similar to Theorem 4.1? In this section we prove the answer is yes. As a consequence of Proposition 3.2 we know it is not possible to do an immediate reduction to Theorem 4.1. The passage from finite sequences to bi-infinite sequences suggests trying a similar passage at the syntactic semigroup level. This motivates the introduction of ζ-semigroups, a generalization of ω-semigroups.

We quickly review here basic definitions about ω-semigroups. For an exhaustive overview, see [19]. An *ω-semigroup* is a two-component algebra $S = (S_+, S_\omega)$ equipped with a binary product on S_+, a mapping from $S_+ \times S_\omega$ into S_ω called the "mixt product", and a surjective mapping $\pi : S_+^\omega \to S_\omega$ called the "infinite product", and such that the following conditions are satisfied:

(1) the set S_+ equipped with its product is a semigroup,
(2) for each s, t in S_+ and u in S_ω, $s(tu) = (st)u$,
(3) for each non-decreasing sequence $(i_n)_{n>0}$ of $\mathbb{N}$ and each sequence $(s_n)_{n\in\mathbb{N}}$ of S_+^ω, $\pi(s_0 s_1 \cdots s_{(i_1-1)},\, s_{i_1} \cdots s_{(i_2-1)}, \cdots) = \pi(s_0,\, s_1,\, s_2, \cdots)$.
(4) for all s in S_+ and for each sequence $(s_n)_{n\in\mathbb{N}}$ of S_+^ω, $s\,\pi(s_0,\, s_1,\, s_2, \cdots) = \pi(s,\, s_0,\, s_1,\, s_2, \cdots)$.

An *ω-semigroup morphism* from $S = (S_+, S_\omega)$ into $T = (T_+, T_\omega)$ is a pair $\varphi = (\varphi_+, \varphi_\omega)$ such that $\varphi_+ : S_+ \to T_+$ is a semigroup morphism and $\varphi_\omega : S_\omega \to T_\omega$ preserves both infinite product and mixt product. The *$\tilde{\omega}$-semigroups* and *$\tilde{\omega}$-semigroup morphisms* are similarly defined, all products operating on the left.

A *ζ-semigroup* is a four-component algebra $S = (S_+, S_\omega, S_{\tilde{\omega}}, S_\zeta)$ such that (S_+, S_ω) is an ω-semigroup, $(S_+, S_{\tilde{\omega}})$ is an $\tilde{\omega}$-semigroup, and with a surjective mapping $\rho : S_{\tilde{\omega}} \times S_\omega \to S_\zeta$ such that if $s \in S_{\tilde{\omega}}$, $t \in S_+$, and $u \in S_\omega$ then $\rho(s,\, tu) = \rho(st,\, u)$. A ζ-semigroup is *finite* if all its four components are finite.

Example 4.2. *Denote by A^ζ the quotient of $A^{\mathbb{Z}}$ under the equivalence relation:*

$$u \sim_\sigma v \Leftrightarrow \exists n \mid u = \sigma^n(v).$$

The algebra $A^\infty = (A^+, A^\omega, A^{\tilde{\omega}}, A^\zeta)$ equipped with the usual concatenation is then a ζ-semigroup, called the free ζ-semigroup *on A.*

Let S and T be ζ-semigroups. A *ζ-semigroup morphism* from S into T is a quadruplet $\varphi = (\varphi_+, \varphi_\omega, \varphi_{\tilde{\omega}}, \varphi_\zeta)$ such that $(\varphi_+, \varphi_\omega)$ (resp. $(\varphi_+, \varphi_{\tilde{\omega}})$) is an ω-semigroup morphism (resp. $\tilde{\omega}$-semigroup morphism), and φ_ζ is a map from S_ζ into T_ζ such that for every s in $S_{\tilde{\omega}}$ and t in S_ω one has $\varphi_\zeta(st) = \varphi_{\tilde{\omega}}(s)\varphi_\omega(t)$. Notice that a ζ-semigroup homomorphism φ is entirely determined by φ_+.

A subset P of A^ζ is *recognized* by a ζ-semigroup homomorphism $\varphi : A^\zeta \to S$ if there exists a subset I of S_ζ such that $P = \varphi_\zeta^{-1}(I)$. We say that P is recognized by a ζ-semigroup S if there is a ζ-semigroup homomorphism $\varphi : A^\zeta \to S$ recognizing P.

Given an element u of A^+, we denote by u^ω the element of A^ω given by the right-infinite product $uuu\ldots$. Dually, $u^{\tilde{\omega}} = \ldots uuuu$. Finally, $u^\zeta = u^{\tilde{\omega}} u^\omega \in A^\zeta$. In absence of confusion we also denote by u^ζ the element $u^{\tilde{\omega}} \cdot u^\omega$ of $A^{\mathbb{Z}}$.

Let P be a subset of A^ζ. The *syntactic congruence on P*, denoted by $\sim_P = (\sim_+, \sim_\omega, \sim_{\tilde\omega}, \sim_\zeta)$, is defined by

$$(1)\ \forall s,t \in A^+, s \sim_+ t \iff \begin{cases} \forall x \in A^{\tilde\omega},\ \forall y \in A^\omega, & xsy \in P \ \Leftrightarrow\ xty \in P \\ \forall x \in A^{\tilde\omega},\ \forall y \in A^+, & x(sy)^\omega \in P \ \Leftrightarrow\ x(ty)^\omega \in P \\ \forall x \in A^+,\ \forall y \in A^\omega, & (xs)^{\tilde\omega} y \in P \ \Leftrightarrow\ (xt)^{\tilde\omega} y \in P \\ \forall x \in A^+, & (xs)^\zeta \in P \Leftrightarrow (xt)^\zeta \in P \end{cases}$$

$$(2)\ \forall s,t \in A^\omega,\quad s \sim_\omega t \iff \big[\, \forall x \in A^{\tilde\omega},\ \ xs \in P \ \Leftrightarrow\ xt \in P \,\big]$$

$$(3)\ \forall s,t \in A^{\tilde\omega},\quad s \sim_{\tilde\omega} t \iff \big[\, \forall x \in A^{\omega},\ \ xs \in P \ \Leftrightarrow\ xt \in P \,\big]$$

$$(4)\ \forall s,t \in A^\zeta,\quad s \sim_\zeta t \iff \big[\, s \in P \ \Leftrightarrow t \in P \,\big]$$

It can be shown that this relation is indeed a congruence of ζ-semigroups on A^∞. Hence we can consider the quotient algebra:

$$\mathcal{S}(P) = A^\infty/{\sim_P} = (A^+/{\sim_+}, A^\omega/{\sim_\omega}, A^{\tilde\omega}/{\sim_{\tilde\omega}}, A^\zeta/{\sim_\zeta}).$$

Proposition 4.3. *If $\mathcal{S}(P)$ is finite then $\mathcal{S}(P)$ is a ζ-semigroup and the quotient map $\pi_P : A^\infty \to \mathcal{S}(P)$ is a ζ-semigroup homomorphism recognizing P.*

We call $\mathcal{S}(P)$ the *syntactic ζ-semigroup of P*, if $\mathcal{S}(P)$ is finite.

Let $\mathcal{X}$ be a subshift of $A^\mathbb{Z}$. Since $\mathcal{X}$ is saturated by the relation $\sim_\sigma$, we do not lose information if we identify $\mathcal{X}$ with $\mathcal{X}/{\sim_\sigma}$. For this reason and for the sake of conciseness, we indistinctly consider $\mathcal{X}$ as a subset of both $A^\mathbb{Z}$ and A^ζ.

Here comes a simple but crucial lemma that links the syntactic ζ-semigroup of a sofic subshift with its classical syntactic semigroup.

Lemma 4.4. *Let $\mathcal{X}$ be a subshift of $A^\mathbb{Z}$. Then $\sim_+$ is the syntactic congruence of $L(\mathcal{X})$. Moreover, $\mathcal{X}$ is sofic if and only if $\mathcal{S}(\mathcal{X})$ is finite.*

Due to very specific properties of the pseudovariety D, it is possible to define the wreath product of a ζ-semigroup with a semigroup in D. This construction was inspired by a similar one by O. Carton on ω-semigroups [10]. Although technical, the following definition merely translates on ζ-semigroups the natural operation of wreath product of automata. Let S be a finite ζ-semigroup, and T a semigroup in D. Let $E(T)$ be the set of idempotents of T. Since $T \in \mathsf{D}$, this set is a subsemigroup of T. The *wreath product* $S \circ T$ is the ζ-semigroup $\big((S_+^{E(T)} \times T), S_\omega^{E(T)}, S_{\tilde\omega} \times E(T), S_\zeta\big)$ defined by:

$$\begin{cases} \forall (f_1,t_1),(f_2,t_2) \in S_+^{E(T)} \times T,\ (f_1,t_1)\cdot(f_2,t_2) = (f, t_1t_2), \text{ with } f(e) = f_1(e)f_2(et_1); \\ \forall (f,t) \in S_+^{E(T)} \times T,\ \forall g \in S_\omega^{E(T)},\ \ (f,t)\cdot g = h, \text{ with } h(e) = f(e)g(et); \\ \forall (s,e) \in S_{\tilde\omega} \times E(T),\ \forall (f,t) \in (S_+^{E(T)} \times T)\,,\ \ (s,e)\cdot(f,t) = (sf(e), et); \\ \forall (s,e) \in S_{\tilde\omega} \times E(T),\ \forall g \in S_\omega^{E(T)},\ \ (s,e)\cdot g = sg(e); \\ \forall (f,t) \in S_+^{E(T)} \times T,\ \ (f,t)^\omega = g, \text{ with } g(e) = f'(e)\big(f'(t')\big)^\omega, \\ \ \text{where } (f',t') \text{ is the idempotent power of } (f,t); \\ \forall (f,t) \in S_+^{E(T)} \times T,\ \ (f,t)^{\tilde\omega} = \big(f'(t')^{\tilde\omega}, t'\big), \ \ \text{where } (f',t') \text{ is the idempotent power of } (f,t). \end{cases}$$

The verification that $S \circ T$ is actually a ζ-semigroup is tedious and similar to the analogous construction of [10]. Note that the semigroup $(S \circ T)_+$ is the homomorphic image of $S_+ \circ T$ by the homomorphism $(f,t) \mapsto (f_{|E(T)}, t)$.

Let $F : A^\mathbb{Z} \to B^\mathbb{Z}$, be a sliding block code. Since F commutes with the shift operation, one can define the function from A^ζ to B^ζ mapping $x/{\sim_\sigma}$ into $F(x)/{\sim_\sigma}$. We also denote such map by F, and call it sliding block code.

Theorem 4.5 ([11, Theorem 2.7]). *Let $F : A^\zeta \to B^\zeta$ be a sliding block code with window size k and let P be a subset of B^ζ recognized by a finite ζ-semigroup Z. Then $F^{-1}(P)$ is recognized by the wreath product $Z \circ \tilde{\mathcal{D}}_{k-1}$.*

Lemma 4.6. *Let P be a subset of A^ζ, and let $\psi : A^\infty \to T$ be a ζ-semigroup homomorphism. If P is recognized by ψ then the set*

$$L(P) = \{u \in P \mid \exists x \in A^{\tilde{\omega}}\, \exists y \in A^\omega : xuy \in P\}$$

is recognized by ψ_+.

Proof. By hypothesis, there is a subset I of T_ζ such that $P = \psi_\zeta^{-1}(I)$. Consider the set

$$I_\psi = \{t \in T_+ \mid \exists x \in A^{\tilde{\omega}}, y \in A^\omega : \psi_{\tilde{\omega}}(x)\, t\, \psi_\omega(y) \in I\}$$

Clearly, since $\psi_{\tilde{\omega}}(x)\, \psi_+(u)\, \psi_\omega(y) = \psi_\zeta(xuy)$ and $\psi_\zeta^{-1}(I) = P$, we have

$$u \in \psi_+^{-1}(I_\psi) \Leftrightarrow \big[\exists x \in A^{\tilde{\omega}}\, \exists y \in A^\omega : \psi_\zeta(xuy) \in I\big] \Leftrightarrow u \in L(P),$$

thus $L(P) = \psi_+^{-1}(I_\psi)$. □

Theorem 4.7. *Let $F : A^\mathbb{Z} \to B^\mathbb{Z}$ be a code with window size k and let $\mathcal{Y}$ be a sofic subshift of $B^\mathbb{Z}$ with syntactic semigroup S. Then the syntactic semigroup of $L(F^{-1}(\mathcal{Y}))$ divides the wreath product $S \circ \tilde{\mathcal{D}}_{k-1}$.*

Proof. Let Z be the syntactic ζ-semigroup of $\mathcal{Y}$, and denote by W the wreath product $Z \circ \tilde{\mathcal{D}}_{k-1}$. By Theorem 4.5 the subshift $F^{-1}(\mathcal{Y})$ is recognized by the ζ-semigroup W. Then by Lemma 4.6 the set $L(F^{-1}(\mathcal{Y}))$ is recognized by the semigroup W_+. Hence, if R is the syntactic semigroup of $F^{-1}(\mathcal{Y})$ then $R \prec W_+$. Since $W_+ \prec Z_+ \circ \tilde{\mathcal{D}}_{k-1}$, and the division between semigroups is a transitive relation, we deduce $R \prec Z_+ \circ \tilde{\mathcal{D}}_{k-1}$. By Lemma 4.4 we have $S = Z_+$. □

5 Classes of sofic subshifts closed under taking divisors

A *full shift* is a subshift of the form $A^\mathbb{Z}$, for some alphabet A. The syntactic semigroup of a language L of A^+ may depend on the alphabet A. For example, the syntactic semigroup of A^+ as a language of A^+ is the trivial semigroup, while if $A \subsetneq B$ then the syntactic semigroup of A^+ as language of B^+ is the unique monoid $\{0,1\}$ with the usual multiplication. The pseudovariety Sl is the least pseudovariety containing this monoid. Hence, to avoid ambiguities, we consider the syntactic semigroup of a full shift to be the trivial semigroup. On the other hand, if $\mathcal{X}$ is a subshift of $A^\mathbb{Z}$ different from a full shift, then there is no risk of ambiguity, because the syntactic semigroup of $L(\mathcal{X})$ is independent of A, basically because all elements of $A^+ \setminus L(\mathcal{X})$ are in the same class of the syntactic congruence [8]. For a pseudovariety V, consider the class $\mathscr{S}(\mathsf{V})$ of the subshifts $\mathcal{X}$ whose syntactic semigroup belongs to V.

Theorem 5.1. *Let V be pseudovariety of semigroups containing Sl. Then the class $\mathscr{S}(\mathsf{V} * \mathsf{D})$ is closed under taking divisors, and in particular it is closed under taking weak equivalent subshifts.*

Proof. Suppose $\mathcal{Y}$ is a subshift of $A^{\mathbb{Z}}$ belonging to $\mathscr{S}(\mathsf{V}*\mathsf{D})$. Let $\mathcal{X}$ be a subshift of $B^{\mathbb{Z}}$ dividing $\mathcal{Y}$. Then there is an integer k and a code $F : A^{\mathbb{Z}} \to {B_\$}^{\mathbb{Z}}$ with window size k such that $\mathcal{X} = F^{-1}(\mathcal{Y})$. Since $\mathsf{SI} \subseteq \mathsf{V}$ the syntactic semigroup of $\mathcal{Y}$ as subshift of ${B_\$}^{\mathbb{Z}}$ also belongs to $\mathsf{V}*\mathsf{D}$. By Theorem 4.7, we have $\mathcal{X} \in \mathscr{S}((\mathsf{V}*\mathsf{D})*\mathsf{D}_{k-1})$. But $\mathsf{V}*\mathsf{D}*\mathsf{D}_{k-1} = \mathsf{V}*\mathsf{D}$, because $\mathsf{D}*\mathsf{D} \subseteq \mathsf{D}$. $\square$

Theorem 5.1 improves a result from [13], where only the relation of shift equivalence was considered. Let $\mathscr{S}_I(\mathsf{V})$ be the class of irreducible subshifts in $\mathscr{S}(\mathsf{V})$. Theorem 5.1 also holds for the operator $\mathscr{S}_I$. For every pseudovariety V of semigroups we have $\mathsf{LV} = \mathsf{LV}*\mathsf{D}$, thus if $\mathsf{SI} \subseteq \mathsf{V}$ then $\mathscr{S}_I(\mathsf{LV})$ is closed under taking weak equivalent irreducible subshifts. There is an infinity of classes of this form [13]. Theorem 5.1 has the following converse:

Theorem 5.2 ([13]). *Let V be pseudovariety of semigroups. Let $\mathscr{O}$ be any of the operators $\mathscr{S}$ or $\mathscr{S}_I$. If $\mathscr{O}(\mathsf{V})$ is closed under taking conjugate subshifts then $\mathsf{LSI} \subseteq \mathsf{V}$ and $\mathscr{O}(\mathsf{V}) = \mathscr{O}(\mathsf{V}*\mathsf{D})$.*

Theorem 5.1 can be used as a method of proving that a certain class of subshifts is closed under division and therefore under weak equivalence. For example, the class of sofic subshifts is the class $\mathscr{S}(\mathsf{S})$, where S is the pseudovariety of all finite semigroups. Hence, an immediate corollary of Theorem 5.1 is that the class of sofic subshifts is closed under divisions. The class of finite type subshifts is also closed under division, but it is not of the form $\mathscr{S}(\mathsf{V})$; on the other hand, the class of *irreducible* finite type subshifts is equal to $\mathscr{S}_I(\mathsf{LCom})$ [13].

Two elements x and y of $A^{\mathbb{Z}}$ are *right-asymptotic* if there is an integer n such that $x_{[n,+\infty[} = y_{[n,+\infty[}$. A code $\varphi : \mathcal{X} \to \mathcal{Y}$ between two subshifts is *left-closing* if distinct right-asymptotic elements of $\mathcal{X}$ have distinct images by φ. For an algorithm to decide whether the cover associated to a labeled graph is left-closing or not see [3]. Clearly one can consider the dual definition of *right-closing* code. A code is *bi-closing* if it is simultaneously right-closing and left-closing. An *almost finite type subshift* is the image of an irreducible finite type subshift by a bi-closing code. Almost finite type subshifts form a class of irreducible sofic subshifts strictly containing the irreducible finite type subshifts. An irreducible sofic subshift is of almost finite type if and only if its right Fischer cover is left-closing [3, Proposition 2.16]. It is known that this class is closed under conjugation [3, Proposition 4.1]. Independently from this result, in [4] it was proved that almost finite type subshifts belong to $\mathscr{S}_I(\mathsf{LInv})$, and the second author proved in [13] that in fact all elements of $\mathscr{S}_I(\mathsf{LInv})$ are almost finite type subshifts. Therefore, since $\mathsf{SI} \subseteq \mathsf{Inv}$, from Theorem 5.1 we deduce the following sharper result:

Theorem 5.3. *The class of almost finite type subshifts is closed under taking irreducible divisors, and hence it is closed under taking weak equivalent irreducible subshifts.*

The class of *aperiodic subshifts* is a class of almost finite type subshifts that deserves some attention [3]. It is proved in [5] that this class is equal to $\mathscr{S}_I(\mathsf{A})$. Since $\mathsf{SI} \subseteq \mathsf{A} = \mathsf{LA}$, Theorem 5.1 also has the following corollary:

Theorem 5.4. *The class of aperiodic subshifts is closed under taking irreducible divisors, and hence it is closed under taking weak equivalent irreducible subshifts.*

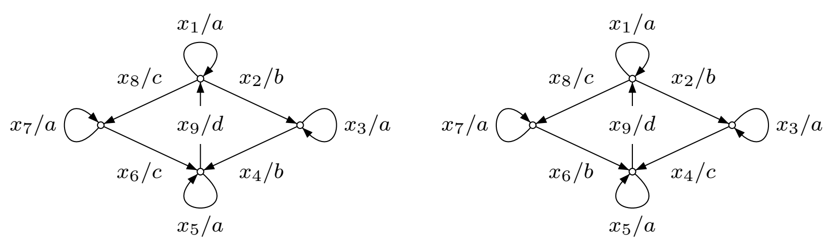
6 Comparison with other invariants

For a code $\varphi : \mathcal{X} \to \mathcal{Y}$, let $\mathcal{M}(\varphi)$ be the set $\{x \in \mathcal{X} : |\varphi^{-1}\varphi(x)| > 1\}$. Clearly, $\sigma(\mathcal{M}(\varphi)) \subseteq \mathcal{M}(\varphi)$ and $\sigma^{-1}(\mathcal{M}(\varphi)) \subseteq \mathcal{M}(\varphi)$. Hence $\overline{\mathcal{M}(\varphi)}$ is a subshift, called *multiplicity subshift of* φ. In general $\mathcal{M}(\varphi)$ is not closed. On the other hand, if $\varphi : \mathcal{X} \to \mathcal{Y}$ is a bi-closing code then $\mathcal{M}(\varphi)$ is closed [9]. One can prove that the multiplicity subshift of a sofic subshift is effectively computable. Note that if (f, g) is a conjugacy between φ and ψ, then $(f_{|\overline{\mathcal{M}(\varphi)}}, g_{|\varphi\overline{\mathcal{M}(\varphi)}})$ is a conjugacy between $\varphi : \overline{\mathcal{M}(\varphi)} \to \varphi\big(\overline{\mathcal{M}(\varphi)}\big)$ and $\psi : \overline{\mathcal{M}(\psi)} \to \psi\big(\overline{\mathcal{M}(\psi)}\big)$.

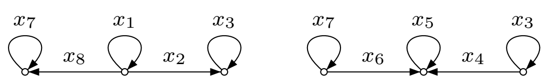
Theorem 6.1 ([9, Theorem 2.8]**).** *Suppose that* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *are mixed subshifts of almost finite type. For each* $i \in \{1, 2\}$*, let* $\pi_i : \mathcal{X}_i \to \mathcal{Y}_i$ *be the right Fischer cover of* $\mathcal{Y}_i$*. Suppose that* $(\pi_1)_{|\mathcal{M}(\pi_1)}$ *and* $(\pi_2)_{|\mathcal{M}(\pi_2)}$ *are conjugate, and that* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are shift equivalent. Then* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *are shift equivalent.*

Since shift equivalence is a very strong conjugacy invariant for sofic subshifts, if $\pi : \mathcal{X} \to \mathcal{Y}$ is the right Fischer cover of a sofic subshift then the conjugacy class of $\pi_{|\mathcal{M}(\pi)}$ together with the shift equivalence class of $\mathcal{X}$ form a conjugacy invariant that is particularly strong when $\mathcal{Y}$ is mixed and of almost finite type.

Consider the sofic subshifts $\mathcal{Y}_1$ and $\mathcal{Y}_2$ with the following corresponding right Fischer covers π_1 and π_2 (the notation x/α means that the edge x is labeled α):



Subshifts $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are mixing almost finite type subshifts with the same zeta function. The domains of the right and left Krieger/Fischer covers are respectively equal. The next invariant to be tested is the multiplicity subshift. The multiplicity subshifts of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are equal to the following subshift $\mathcal{X}$:



To prove that $\pi_{1|\mathcal{X}}$ is not conjugate with $\pi_{2|\mathcal{X}}$ we are going to use the following lemma:

Lemma 6.2 ([9, Lema 2.3]**).** *Let* $\varphi : \mathcal{X} \to \mathcal{Y}$ *and* $\psi : \mathcal{X} \to \mathcal{Z}$ *be codes with equal domain. Then* φ *and* ψ *are conjugate if and only if there is for* $\mathcal{X}$ *an automorphism*[1] F *such that* $\psi \circ F$ *and* φ *have the same kernel*[2]*.*

Let F be an automorphism of $\mathcal{X}$, with block map f with window size n. Since F permutes fixed points, there is $i \in \{1, 3, 5, 7\}$ such that $F(x_i^\zeta) = x_1^\zeta$. Suppose $i \neq 1$. Then there are k, j such that $k \neq i$ and $x_k^{\tilde{\omega}}.x_j x_i^\omega \in \mathcal{X}$. Since $f(x_i^n) = x_1$, we have $F(x_k^{\tilde{\omega}}.x_j x_i^\omega) \sim_\sigma x_r^{\tilde{\omega}}.x_s x_1^\omega$. Since $x_l x_1 \in L(\mathcal{X})$ implies $l = 1$, we have $F(x_k^{\tilde{\omega}}.x_j x_i^\omega) = x_1^\zeta = F(x_i^\zeta)$, contradicting F being one-to-one. Hence

[1]An *automorphism* for $\mathcal{X}$ is a conjugacy from $\mathcal{X}$ to $\mathcal{X}$.

[2]Recall that the kernel of a map $h : P \to Q$ is the set $\{(x, y) \in P \times P \mid h(x) = h(y)\}$.

$F(x_1^\zeta) = x_1^\zeta$. Analogously, $F(x_5^\zeta) = x_5^\zeta$, thus $\{F(x_3^\zeta), F(x_7^\zeta)\} = \{x_3^\zeta, x_7^\zeta\}$. Let $z = x_1^{\tilde{\omega}}.x_2x_3^\omega$ and $t = x_3^{\tilde{\omega}}.x_4x_5^\omega$. Then $z, t \in \mathcal{X}$ and $\pi_1(z) = \pi_1(t) = a^{\tilde{\omega}}.ba^\omega$. Suppose $F(x_3^\zeta) = x_7^\zeta$ and $F(x_7^\zeta) = x_3^\zeta$. Then $F(z) \sim_\sigma x_1^{\tilde{\omega}}.x_8x_7^\omega$ and $F(t) \sim_\sigma x_7^{\tilde{\omega}}.x_6x_5^\omega$, thus $\pi_2F(z) \sim_\sigma a^{\tilde{\omega}}.ca^\omega$ and $\pi_2F(t) \sim_\sigma a^{\tilde{\omega}}.ba^\omega$. In particular, $\pi_2F(z) \neq \pi_2F(t)$, and the same conclusion holds if $F(x_3^\zeta) = x_3^\zeta$ and $F(x_7^\zeta) = x_7^\zeta$. Therefore $\pi_{1|\mathcal{X}}$ and $\pi_{2|\mathcal{X}}$ are not conjugate, by Lema 6.2. Hence $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are not conjugate.

The preceding arguments are somewhat ad-hoc, and depend on the knowledge of the group of automorphisms of a subshift, a difficult problem in general: we do not know if the automorphisms group of the two-letter and the three-letter full shifts are isomorphic [18, Chapter 13].

On the other hand, as observed in [13], for the pseudovariety V of finite semigroups satisfying the identity $x^3 = x^2$, one has $\mathcal{X} \notin \mathscr{S}(\mathsf{LV})$ and $\mathcal{Y} \in \mathscr{S}(\mathsf{LV})$. Since $\mathsf{Sl} \subseteq \mathsf{V}$, by Theorem 5.1 we conclude that $\mathcal{X}$ and $\mathcal{Y}$ are not weak equivalent. Hence Theorem 5.1 provides an expedite form of proving not only that $\mathcal{X}$ and $\mathcal{Y}$ are not conjugate, but also that they are far from being conjugate, in the sense that weak equivalence is considered a very weak conjugacy invariant.

7 A topological proof

In this section we use a different method for proving Theorem 5.1, based on profinite semigroup theory. As an introductory reference for this theory see [2].

A semigroup endowed with a compact topology for which the multiplication is continuous is called a *compact semigroup*. We consider finite semigroups as compact semigroups, endowing them with the discrete topology. A compact semigroup S is said to be *generated* by a map $\iota : A \to S$ if the subsemigroup of S generated by the image of A is dense in S. Let V be a pseudovariety of semigroups. A *pro*-V semigroup is a projective limit of semigroups from V. A *pro*-V semigroup is therefore a compact semigroup. For the pseudovariety S of all finite semigroups, the term *profinite* is usually used instead of pro-S. For every set A there is a pro-V semigroup such that $\overline{\Omega}_A\mathsf{V}$ is generated by a map ι with domain A with the property that for every map φ from A into a semigroup S from V there is a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. The semigroup $\overline{\Omega}_A\mathsf{V}$ is the unique pro-V semigroup with these properties, up to isomorphism of compact semigroups. For this reason it is called the *free profinite semigroup relatively to* V (or *free pro*-V *semigroup*) generated by A. Assuming A is finite (as we do from here on), the topology of $\overline{\Omega}_A\mathsf{V}$ is generated by a metric. If V contains some non-trivial semigroup, then ι is injective, thus A is considered as a subset of $\overline{\Omega}_A\mathsf{V}$. And if V contains N then A^+ embeds in $\overline{\Omega}_A\mathsf{V}$ as a dense subsemigroup whose elements are isolated points.

The following proposition makes the connection between the combinatorial properties of V-recognizable languages and the topology of $\overline{\Omega}_A\mathsf{V}$, when $\mathsf{N} \subseteq \mathsf{V}$. For a more general result see [1, Theorem 3.6.1], or [2, Section 3].

Proposition 7.1. *Let* V *be a pseudovariety containing* N. *If* L *is a language of* A^+ *then* L *is* V*-recognizable if and only if the closure of* L *in* $\overline{\Omega}_A\mathsf{V}$ *open. The topology of* $\overline{\Omega}_A\mathsf{V}$ *is generated by the closures of the* V*-recognizable subsets of* A^+.

Theorem 7.2. *Let* V *be a pseudovariety containing* Sl *and* N. *For every alphabet* A *and non-negative integer* k, *the map* $\Phi_k : A^+ \to (A^{k+1})^*$ *has a unique continuous extension from* $\Omega_A(\mathsf{V} * \mathsf{D}_k)$ *to* $(\overline{\Omega}_{A^k}\mathsf{V})^1$, *which we denote by* Φ_k^{V}.

Theorem 7.2 was proved by Almeida for the pseudovariety of all finite semigroups [1, Lemma 10.6.11]; the more general case stated here can be easily deduced from another result of Almeida [1, Theorem 10.6.12].

Theorem 7.3. *Let* V *be a pseudovariety containing* Sl *and* N. *Consider a code* $F : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ *with window size* k. *Let* $\mathcal{Y}$ *be a subshift of* $B^{\mathbb{Z}}$. *If* $\mathcal{Y} \in \mathscr{S}(\mathsf{V})$ *then* $F^{-1}(\mathcal{Y}) \in \mathscr{S}(\mathsf{V} * \mathsf{D}_{k-1})$.

Proof. Let $f : A^k \to B$ be a block map for F. Then there is a unique continuous homomorphism $\hat{f} : \overline{\Omega}_{A^k}\mathsf{V} \to \overline{\Omega}_B\mathsf{V}$ extending f. Denote by W the pseudovariety $\mathsf{V} * \mathsf{D}_{k-1}$. Let $\tilde{f} : \overline{\Omega}_A\mathsf{W} \to (\overline{\Omega}_B\mathsf{V})^1$ be the map $\hat{f} \circ \Phi_k^{\mathsf{V}}$. The map $\tilde{f}$ is a (unique) continuous extension of f.

Let $\mathcal{X} = F^{-1}(\mathcal{Y})$. Since $\mathcal{Y} \in \mathscr{S}(\mathsf{V})$, the set $\overline{L(\mathcal{Y})}$ is open in $\overline{\Omega}_B\mathsf{V}$, by Proposition 7.1. By the same proposition, what we want to prove is that $\overline{L(\mathcal{X})}$ is open in $\overline{\Omega}_A\mathsf{W}$. Let $u \in \overline{L(\mathcal{X})}$. Since $L(\mathcal{X})$ is a prolongable language, with a simple compactness argument [12] one proves that for every integer l there are finite words r_l and s_l with length greater that l such that $r_l u s_l \in \overline{L(\mathcal{X})}$. Since $\overline{\Omega}_A\mathsf{W}$ is compact, taking subsequences if necessary, we can assume that $(r_l)_l$ and $(s_l)_l$ converge to some elements r and s of $\overline{\Omega}_A\mathsf{W}$, respectively. Then $rus \in \overline{L(\mathcal{X})}$. Since $\tilde{f}(L(\mathcal{X})) \subseteq L(\mathcal{Y}) \cup \{1\}$ and $\tilde{f}$ is continuous, one has $\tilde{f}(rus) \in \overline{L(\mathcal{Y})}$. Let $(u_n)_n$ be an arbitrary sequence of elements of A^+ converging to u. Then $\lim_{n\to+\infty, l\to+\infty} \tilde{f}(r_l u_n s_l) = \tilde{f}(rus)$. Since $\overline{L(\mathcal{Y})}$ is an open neighborhood of $\tilde{f}(rus)$, there is an integer N such that if $n, l > N$ then $\tilde{f}(r_l u_n s_l) \in \overline{L(\mathcal{Y})}$. Let $n > N$. Since the elements of B^+ are isolated in $\overline{\Omega}_B\mathsf{V}$, we have $\tilde{f}(r_l u_n s_l) \in L(\mathcal{Y})$ for all $l > N$. Consider arbitrary elements $p_l \in A^{\tilde{\omega}}$, $q_l \in A^{\omega}$ and let $x_{n,l} = p_l r_l . u_n s_l q_l \in A^{\mathbb{Z}}$. Let x_n be an adherent point of $(x_{n,l})_l$ in $A^{\mathbb{Z}}$. Then, given a positive integer k, for sufficiently large l the word $F(x_n)_{[-k,k]}$ is a factor of $\tilde{f}(r_l u_n s_l)$, hence it belongs to $L(\mathcal{Y})$. Since k is arbitrary, we have $F(x_n) \in \mathcal{Y}$, thus $x_n \in \mathcal{X}$. Since u_n is a factor of x_n, in particular, $u_n \in L(\mathcal{X})$. Since $(u_n)_n$ is an arbitrary sequence of elements of A^+ converging to u and A^+ is dense in $\overline{\Omega}_A\mathsf{W}$, we conclude that $\overline{L(\mathcal{X})}$ is open. $\square$

Since $\mathsf{V} * \mathsf{D} = (\mathsf{V} * \mathsf{D}) * \mathsf{D}_k$, and $\mathsf{Sl}, \mathsf{N} \subseteq \mathsf{LSl} = \mathsf{Sl} * \mathsf{D} \subseteq \mathsf{V} * \mathsf{D}$, Theorem 5.1 is a corollary of Theorem 7.3.

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