A Burnside Approach to the Termination of Mohri's Algorithm for Polynomially Ambiguous Min-Plus-Automata*

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Abstract. We show that the termination of Mohri's algorithm is decidable for polynomially ambiguous weighted finite automata over the tropical semiring which gives a partial answer to a question by Mohri from 1997 [18]. The proof relies on an improvement of the notion of the twins property and a Burnside type characterization for the finiteness of the set of states produced by Mohri's algorithm.

1 Introduction

Weighted finite automata over the tropical semiring (for short WFA) are of great theoretical and practical interest in computer science, e.g., [5,6,11,13,17,18].

To achieve efficient implementations, one is interested in utilizing subsequential (deterministic) WFA [18]. In contrast to unweighted automata, there are WFA which do not admit a subsequential equivalent. Mohri developed an algorithm to determinize WFA [18], which is implemented within the AT&T FSM Library ™. There are WFA on which Mohri's algorithm does not terminate despite there are subsequential equivalents. Nevertheless, his algorithm is very successful on WFA which occur in speech recognition.

Mohri raised the question whether it is decidable whether his algorithm terminates on a given WFA [1,18]. For trim, unambiguous WFA, he utilized the twins property to give a decidable characterization of the WFA on which his algorithm terminates [18]. In general, Mohri's question remains open.

In this paper, we give a partial answer to Mohri's question: we show that the termination of his algorithm is decidable for polynomially ambiguous WFA. A WFA is *polynomially ambiguous* if the number of accepting paths (computations) for some word w is polynomially bounded in the length of w.

We examine some particular WFA to understand the inadequacy of the twins property and develop the notion of the *clones property*. For trim, finitely ambiguous WFA the clones property coincides with the twins property. The main result of the paper states that the clones property is a decidable, sufficient, and necessary condition for the termination of Mohri's algorithm on trim, polynomially ambiguous WFA (Theorem 4, Corollary 7). To prove that the clones

* See www.math.tu-dresden.de/k̃irsten/ for a long version [12].

property is sufficient, we need some involved algebraic tools as Simon's factorization forest theorem and a Burnside type characterization of the finiteness of the set of states produced by Mohri's algorithm. This Burnside type characterization states that Mohri's algorithm terminates on trim, polynomially ambiguous WFA iff it terminates on every sequence of the form $(vw^k)_{k\geq 1}$.

2 Notations

Let $\mathbb{N} = \{0, 1, \dots\}$. We denote by $|M|$ the number of elements of a finite set M. For a semigroup S, let $\mathsf{E}(S) := \{e \in S \mid ee = e\}$ be the set of *idempotents* of S.

The *Boolean semiring* $(\mathbb{B}, \vee, \wedge, 1, 0)$ consists of the set $\mathbb{B} = \{1, 0\}$ whereas $\vee$ (resp. $\wedge$) are understood as logical or (resp. and) and 1 (resp. 0) are understood as true (resp. false). The *tropical semiring* $(\mathbb{Z}, \min, +, 0, \infty)$ consists of the set $\mathbb{Z} := \{\dots, -1, 0, 1, \dots, \infty\}$ whereas min and $+$ are understood as usual. The mapping $\alpha : \mathbb{Z} \to \mathbb{B}$ defined by $\alpha(\infty) = 0$ and $\alpha(z) = 1$ for $z \in \mathbb{Z} \setminus \{\infty\}$ is a homomorphism. We extend α componentwise to matrices.

Let Q be a finite set. Let $B, B' \in \mathbb{Z}^Q$ and $A \in \mathbb{Z}^{Q\times Q}$. We understand BA as a product of a $1 \times Q$-matrix and a $Q \times Q$-matrix, and we understand AB' as a product of a $Q \times Q$-matrix and a $Q \times 1$-matrix. Hence, $BAB' \in \mathbb{Z}$.

We define $\oplus : \mathbb{Z}\times\mathbb{Z}^Q \to \mathbb{Z}^Q$ by setting for $z \in \mathbb{Z}$, $B \in \mathbb{Z}^Q$, $i \in Q$, $(z\oplus B)[i] := z + B[i]$. We have for $z, z' \in \mathbb{Z}$, $B \in \mathbb{Z}^Q$, $A \in \mathbb{Z}^{Q\times Q}$, $(z \oplus B)A = z \oplus (BA)$ and $z \oplus (z' \oplus B) = (z + z') \oplus B$ which allows us to write $z \oplus BA$ resp. $z \oplus z' \oplus B$.

We identify the members of $\mathbb{B}^Q$ with subsets of Q. For example, for $C \subseteq Q$ and $A \in \mathbb{B}^{Q\times Q}$, we can write CA, and we have $CA \subseteq Q$ but also $CA \in \mathbb{B}^Q$.

Let Σ be a finite set of symbols. We denote by Σ^* the free monoid over Σ. We denote the empty word by ε. We denote $\Sigma^+ := \Sigma^* \setminus \{\varepsilon\}$. For every $w \in \Sigma^*$, we denote by $|w|$ the length of w. We call a word u a *factor* (resp. *prefix*) of a word w if $w \in \Sigma^* u \Sigma^*$ (resp. $w \in u\Sigma^*$).

3 Overview

3.1 Weighted Finite Automata

A *weighted finite automaton over* $\mathbb{Z}$ (for short *WFA*) is a tuple $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ whereas Q is a non-empty, finite set of *states*, $\theta : \Sigma^* \to \mathbb{Z}^{Q\times Q}$ is a homomorphism, and $\lambda, \varrho \in \mathbb{Z}^Q$. Let $\mathcal{A}$ be a WFA over $\mathbb{Z}$. It computes a mapping $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}$ by $|\mathcal{A}|(w) := \lambda\theta(w)\varrho$ for $w \in \Sigma^*$. The mappings computed by WFA are often called *recognizable formal power series*. For an overview on formal power series, the reader is referred to [2,19].

We call two WFA $\mathcal{A}_1$ and $\mathcal{A}_2$ *equivalent* iff they compute the same mapping.

We call a state $q \in Q$ *accessible* if there are words $u, v \in \Sigma^*$ such that $(\lambda\theta(u))[q] \neq \infty$ and $(\theta(v)\varrho)[q] \neq \infty$. We call $\mathcal{A}$ *trim* if every $q \in Q$ is accessible. Given some WFA $\mathcal{A}$, one can compute the set of accessible states and reduce $\mathcal{A}$ to an equivalent trim WFA.

Let $I := \{\, q \in Q \mid \lambda[q] \neq \infty \,\}$ and $F := \{\, q \in Q \mid \varrho[q] \neq \infty \,\}$.

Let $p, q \in Q$ and $a \in \Sigma$. If $\theta(a)[p, q] \neq \infty$, then we call $\big(p,\, a,\, \theta(a)[p, q],\, q\big)$ a

transition. Let $m \geq 0$ and $\pi = (q_0, a_1, k_1, q_1)(q_1, a_2, k_2, q_2) \dots (q_{m-1}, a_m, k_m, q_m)$ be a sequence of transitions. We call π a *path from* q_0 *to* q_m or for short a *path*. We call $a_1 \dots a_m$ the *label of* π. We call π *accepting* if $q_0 \in I$ and $q_m \in F$.

Let $p, q \in Q$ and $w \in \Sigma^*$. We denote by $p \overset{w}{\rightsquigarrow} q$ the set of all paths from p to q which are labeled by w. For $R, R' \subseteq Q$, we denote by $R \overset{w}{\rightsquigarrow} R'$ the union of $r \overset{w}{\rightsquigarrow} r'$ for every $r \in R$, $r' \in R'$.

Let $k \geq 1$. If for every $w \in \Sigma^*$, there are at most k paths in $I \overset{w}{\rightsquigarrow} F$, then we call $\mathcal{A}$ *k-ambiguous*. If $\mathcal{A}$ is 1-ambiguous, then we call $\mathcal{A}$ *unambiguous*. If $\mathcal{A}$ is k-ambiguous for some $k \geq 1$, then we call $\mathcal{A}$ *finitely ambiguous*.

The classes of mappings which are computable by k-ambiguous WFA for $k = 1, 2, \dots$ form a strict hierarchy which exhausts the class of mappings which are computable by finitely ambiguous WFA. The latter class is strictly included in the class of all recognizable formal power series over $\mathbb{Z}$. See [14] for the strictness of these inclusions and other important subclasses of WFA.

Let $P : \mathbb{N} \to \mathbb{N}$ be some polynomial. If for every $w \in \Sigma^*$, there are at most $P(|w|)$ paths in $I \overset{w}{\rightsquigarrow} F$, then we call $\mathcal{A}$ *polynomially ambiguous*. Polynomially ambiguous (unweighted) automata have been studied by various authors, e.g., [9,10,16]. The following characterization is used implicitly in [9,10] (cf. Proof of Theorem 3.1 in [10] or Lemma 4.3 in [9]).

Theorem 1. *A trim WFA $\mathcal{A}$ is polynomially ambiguous iff for every state q and every $w \in \Sigma^*$, there is at most one path in $q \overset{w}{\rightsquigarrow} q$.*

Thus, cycle-unambiguous WFA [1] coincide with polynomially ambiguous WFA.

It is undecidable whether two given WFA over $\mathbb{Z}$ are equivalent [15]. One can deduce from [15] that the same problem is undecidable for polynomially ambiguous WFA. The equivalence of finitely ambiguous WFA is decidable [8].

3.2 Mohri's algorithm

For some background and formal definitions of subsequential WFA, the reader is referred to [12]. For run-time efficiency, one is interested in implementing subsequential WFA rather than non-subsequential WFA [18]. In [18], Mohri presented an algorithm which tries to transform a given WFA into an equivalent, subsequential WFA. If the algorithm terminates, then it constructs an equivalent, subsequential WFA. However, the algorithm does not always terminate, even if an equivalent, subsequential WFA does exist.

Let $\mathcal{A} = [Q, E, \lambda, \varrho]$ be a WFA over $\mathbb{Z}$. Let $n := |Q|$ and $Q = \{1, \dots, n\}$. In this short version, we just explain how Mohri's algorithm constructs the set of states for an equivalent, subsequential WFA, because finiteness of the set of states is the crucial point for the termination.

For $B \in \mathbb{Z}^n$, let $\min(B) := \min_{1 \leq i \leq n} B[i]$. For $B \in \mathbb{Z}^n \setminus \{(\infty, \dots, \infty)\}$, let $\mathsf{nf}(B) := (-\min(B)) \oplus B$ and $\mathsf{nf}\big((\infty, \dots, \infty)\big) = (\infty, \dots, \infty)$.

Mohri's algorithm utilizes the set $Q' := \{\mathsf{nf}(\lambda\theta(w)) \mid w \in \Sigma^*\}$ as states for an equivalent, subsequential WFA. It terminates iff Q' is a finite.

Let $(w_k)_{k \geq 1}$ be some sequence of words in Σ^*. We say that Mohri's *algorithm terminates on* $(w_k)_{k \geq 1}$ *on* $\mathcal{A}$ if the set $\{\mathsf{nf}(\lambda\theta(w_k)) \mid k \geq 1\}$ is finite.

3.3 On the Twins Property

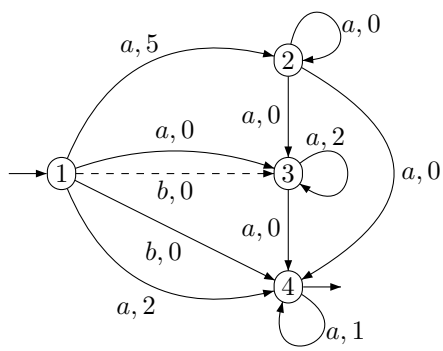
The twins property was introduced by Choffrut in 1977 [4] in the framework of string-to-string transducers. In 1997 [18,1], Mohri generalized the twins property to WFA. Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a WFA. Two states $q, q' \in Q$ are called *siblings* if there exists some $u \in \Sigma^*$ such that $\lambda\theta(u)[q] \neq \infty$ and $\lambda\theta(u)[q'] \neq \infty$. Two siblings $q, q' \in Q$ are called *twins* if they satisfy the following condition:

TW. For every $v \in \Sigma^*$ satisfying $\theta(v)[q, q] \neq \infty$ and $\theta(v)[q', q'] \neq \infty$, we have $\theta(v)[q, q] = \theta(v)[q', q']$.

The WFA $\mathcal{A}$ has the *twins property* iff every siblings are twins. In [18], it is shown that the twins property is sufficient for the termination of Mohri's algorithm. The main weakness the twins property is that the twins property is not necessary for the termination of Mohri's algorithm. However, we have

Theorem 2 ([18, Theorem 12]). *Let $\mathcal{A}$ be a trim, unambiguous WFA.* Mohri*'s algorithm terminates on $\mathcal{A}$ iff $\mathcal{A}$ satisfies the twins property.*

Example 3. Let $\Sigma = \{a, b\}$. We consider the WFA $\mathcal{A}_1 = [Q, \theta_1, \lambda, \varrho]$ and $\mathcal{A}_2 = [Q, \theta_2, \lambda, \varrho]$ shown below whereas $\lambda = (0, \infty, \infty, \infty)$ and $\varrho = (\infty, \infty, \infty, 0)$. The only difference between $\mathcal{A}_1$ and $\mathcal{A}_2$ is that the dashed transition between state 1 and 3 does not exist in $\mathcal{A}_1$ but it does in $\mathcal{A}_2$.



By Theorem 1, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are polynomially ambiguous.

Both $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same siblings: $(1, 1)$ and $\{2, 3, 4\} \times \{2, 3, 4\}$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ do not satisfy the twins property, e.g., $\theta_i(a)[2, 2] = 0 \neq 2 = \theta_i(a)[3, 3]$ for $i \in \{1, 2\}$. Mohri's algorithm terminates on $\mathcal{A}_1$. It does not terminate on $\mathcal{A}_2$ because it produces infinitely many states on the sequence $(ba^k)_{k \geq 1}$.

The key question is how to define a variant of the twins property which allows to distinguish between $\mathcal{A}_1$ and $\mathcal{A}_2$. Let us try an approach which relies on some comparison of siblings, i.e., we try to establish some condition (TW') which is similar to the above condition (TW), and we define that some WFA satisfies the (TW')-twins property if every siblings satisfy (TW').

Consider the siblings $(2, 3)$. For every $p \in Q$, $w \in \Sigma^*$, we have $\theta_1(w)[2, p] = \theta_2(w)[2, p]$ and $\theta_1(w)[3, p] = \theta_2(w)[3, p]$. If (TW') is somehow defined by a comparison of siblings, then $(2, 3)$ satisfies (TW') in $\mathcal{A}_1$ iff $(2, 3)$ satisfies (TW') in $\mathcal{A}_2$. Thus, (TW') cannot distinguish between $(2, 3)$ in $\mathcal{A}_1$ and $(2, 3)$ in $\mathcal{A}_2$.

The same effect happens for every pair of siblings in $\{2,3,4\}\times\{2,3,4\}$. There is one other pair of siblings: $(1,1)$. If (TW') is somehow defined by a comparison of siblings, then (TW') should be satisfied for sibling pairs of the form (q,q) since it means to compare a state to itself.

As a conclusion, it seems to be impossible to define (TW') in a way that $\mathcal{A}_1$ satisfies the (TW')-twins property but $\mathcal{A}_2$ does not. □

3.4 Main Results

Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a WFA. Set $n := |Q|$ and assume $Q = \{1, \ldots, n\}$.

We call some $C \subseteq Q$ a *clone* if there is some $w \in \Sigma^*$ such that $C = \{q \in Q \mid \lambda\theta(w)[q] \neq \infty\}$. We denote the set of all clones of $\mathcal{A}$ by $\mathsf{Clones}(\mathcal{A}) \subseteq 2^Q$.

Let $C \subseteq Q$, $A \in \mathbb{Z}^{n\times n}$, and assume $\alpha(A) \in \mathsf{E}(\mathbb{B}^{n\times n})$. We say that C is *stable* on A if $C\alpha(A) = C$. Assume that C is stable on A. Let $q \in C$. We say that q *has a minimal cycle in* C *and* A if $A[q,q] = \min\{A[p,p] \mid p \in C\}$. We say that C and A have the *clones property* if for every $p \in Q$ satisfying $A[p,p] \neq \infty$, there is some $q \in C$ such that q has a minimal cycle in C and $A[q,p] \neq \infty$.

We say that $\mathcal{A}$ has the *clones property* if for every $C \in \mathsf{Clones}(\mathcal{A})$ and every $w \in \Sigma^*$, C and $\theta(w)$ have the clones property, provided that $\alpha(\theta(w)) \in \mathsf{E}(\mathbb{B}^{n\times n})$ and $\theta(w)$ is stable on C. Our main result is the following theorem:

Theorem 4. *Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a trim, polynomially ambiguous WFA. The following assertions are equivalent:*

1. Mohri*'s algorithm terminates on $\mathcal{A}$.*
2. *For every $v, w \in \Sigma^*$,* Mohri*'s algorithm terminates on $(vw^k)_{k\geq 1}$ on $\mathcal{A}$.*
3. *The WFA $\mathcal{A}$ satisfies the clones property.*

Note that (1) ⇒ (2) in Theorem 4 is obvious.

Example 1 (continued). We continue the examination of $\mathcal{A}_1$ and $\mathcal{A}_2$. Mohri's algorithm does not terminate on the sequence $(ba^k)_{k\geq 1}$ on $\mathcal{A}_2$. We utilize this sequence to show that $\mathcal{A}_2$ does not satisfy the clones property. The clone $C := \{q \in Q \mid \lambda\theta_2(b)[q] \neq \infty\} = \{3,4\}$ is stable on $\theta_2(a)$ and $\alpha(\theta_2(a)) \in \mathsf{E}(\mathbb{B}^{4\times 4})$. Since, $\theta_2(a)[3,3] = 2$ and $\theta_2(a)[4,4] = 1$, only the state 4 has a minimal cycle in C and $\theta_2(a)$. We have $\theta_2(a)[4,3] = \infty$. Consequently, C and $\theta_2(a)$ do not have the clones property, and hence, $\mathcal{A}_2$ does not satisfy the clones property.

Since $C \notin \mathsf{Clones}(\mathcal{A}_1)$, we cannot use the same argument to show that $\mathcal{A}_1$ does not satisfy the clones property. Indeed, Mohri's algorithm terminates on $\mathcal{A}_1$, and by Theorem 4, $\mathcal{A}_1$ satisfies the clones property. □

We show some connections between the clones property and the twins property.

Theorem 5. *If a WFA has the twins property, then it has the clones property.*

Proof (sketch). Let $C \in \mathsf{Clones}(\mathcal{A})$ and $w \in \Sigma^*$ such that $\alpha(\theta(w)) \in \mathsf{E}(\mathbb{B}^{n\times n})$ and $\alpha(\theta(w))$ is stable on C. The key idea is that by the twins property, every $p \in C$ which satisfies $\theta(w)[p,p] \neq \infty$ has a minimal cycle in C and $\theta(w)$.

Theorem 6. *Let $\mathcal{A} = [Q, \theta, \lambda, \varrho]$ be a trim, finitely ambiguous WFA over $\mathbb{Z}$. The following assertions are equivalent:*

1. *The WFA $\mathcal{A}$ satisfies the clones property.*
2. *The WFA $\mathcal{A}$ satisfies the twins property.*
3. MOHRI*'s algorithm terminates on $\mathcal{A}$.*

Proof (sketch). We have (1)⇔(3) and (2)⇒(1) by Theorem 4 and 5. To show (1)⇒(2), let q, q' be siblings and $w \in \Sigma^*$ such that $\theta(w)[i,i] \neq \infty$ for $i \in \{q, q'\}$. Let $C \in \mathsf{Clones}(\mathcal{A})$ such that $q, q' \in C$. Assume $\alpha(\theta(w)) \in \mathsf{E}(\mathbb{B}^{Q\times Q})$ and C is stable on $\theta(w)$. Since, $\mathcal{A}$ is finitely ambiguous and $\alpha(\theta(w)) \in \mathsf{E}(\mathbb{B}^{Q\times Q})$, we can deduce that for $p \neq p'$ satisfying $\theta(w)[p,p] \neq \infty$ and $\theta(w)[p',p'] \neq \infty$, we have $\theta(w)[p,p'] = \infty$. Hence, by the clones property, both q and q' have a minimal cycle in C and $\theta(w)$, i.e., $\theta(w)[q,q] = \theta(w)[q',q']$.

If $\alpha(\theta(w)) \notin \mathsf{E}(\mathbb{B}^{Q\times Q})$ or if C is not stable on $\theta(w)$, then we can use some $k \geq 1$ such that $\alpha(\theta(w^k)) \in \mathsf{E}(\mathbb{B}^{Q\times Q})$ [12].

Corollary 7. *There is an algorithm which decides whether* MOHRI*'s algorithm terminates on a given trim, polynomially ambiguous WFA $\mathcal{A}$.*

Proof. The algorithm runs two simultaneous processes. One process is MOHRI's algorithm on $\mathcal{A}$. The other one searches for a clone and a word which violate the clones property. By Theorem 4, exactly one of the processes terminates.

The restriction to trim WFA in our main results is not a problem. Every WFA $\mathcal{A}$ can be reduced in polynomial time to an equivalent trim WFA $\mathcal{A}'$. If MOHRI's algorithm terminates on $\mathcal{A}$, then it terminates on $\mathcal{A}'$ [12]. Henceforth, there is no motivation to apply MOHRI's algorithm to WFA which are not trim.

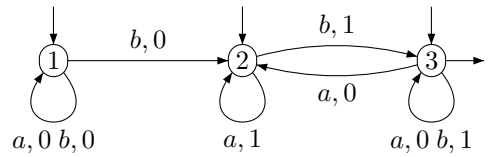
3.5 Conclusions and Open Questions

We can decide whether MOHRI's algorithm terminates on a given polynomially ambiguous WFA which is a partial answer to a question from [18,1]. It is quite interesting to have a decidability result for a class of WFA for which the equivalence problem is undecidable [15]. The equivalence (2)⇔(3) in Theorem 6 generalizes Theorem 2 by MOHRI to finitely unambiguous WFA.

In the tropical semiring, the twins property was a suitable concept just for unambiguous WFA. By introducing the clones property, we came over the inadequacies of the twins property for polynomially ambiguous WFA. Remarkably, the twins and the clones property coincide for finitely ambiguous WFA.

It raises the question whether one can generalize our results to trim WFA which are not necessarily polynomially ambiguous. By the following example, (2) ⇒ (1) in Theorem 4 is not true for arbitrary WFA.

Example 8. We examine $\mathcal{A}$ shown below where $\lambda = (0,0,0)$, $\varrho = (\infty, \infty, 0)$.



Since, there are two different cycles for ab at state 3, $\mathcal{A}$ is not polynomially ambiguous. For $w \in \Sigma^*$, we have $\lambda\theta(w)[1] = 0$, $\lambda\theta(w)[2] \geq 0$, and $\lambda\theta(w)[3] \geq 0$. Hence, we have $\min(\lambda\theta(w)) = 0$ and $\mathsf{nf}(\lambda\theta(w)) = \lambda\theta(w)$.

We show that $\mathcal{A}$ satisfies (2) in Theorem 4. Let $v, w \in \Sigma^*$.

Assume $w \in a^+$. For large k, we have $\lambda\theta(vw^k)[2] = \lambda\theta(vw^k)[3] = \lambda\theta(vw)[3]$.

Assume $w \in \Sigma^* b \Sigma^*$. Let $\ell \geq 0$ such that a^ℓ is the longest factor of the form a^* in words vw^+. By an induction on prefixes of vw^+, we can show $0 \leq \lambda\theta(vw^k)[2] \leq \ell$ and $0 \leq \lambda\theta(vw^k)[3] \leq \ell + 1$ for every $k \geq 1$.

Thus, for every $v, w \in \Sigma^*$, Mohri's algorithm terminates on $(vw^k)_{k \geq 1}$.

Let $w_1 := ba$ and $w_{k+1} := w_k b a^{k+1}$ for $k \geq 1$. We have $\lambda\theta(w_k) = (0, k, k)$ for $k \geq 1$, i.e., Mohri's algorithm does not terminate on $\mathcal{A}$. □

It is an open question whether one can achieve a characterization similar to Theorem 4 for arbitrary WFA by a nested pumping technique using Hashiguchi's k-expressions [7]. Another open problem is to develop a practical algorithm to decide the clones property of polynomially ambiguous WFA.

4 The Main Proofs

4.1 On Boolean Matrices

Let $n \geq 1$ for this section. Let $e \in \mathsf{E}(\mathbb{B}_{n \times n})$. We associate a binary relation $\leq_e$ on $\{1, \ldots, n\}$ to e by setting $i \leq_e j$ iff $e[i, j] = 1$. The relation $\leq_e$ is transitive.

Let S be a subsemigroup of $\mathbb{B}^{n \times n}$ for the rest of this section. We call S *polynomially ambiguous* if there is some polynomial $P : \mathbb{N} \to \mathbb{N}$ such that for every $k \geq 1$, $p_1, \ldots, p_k \in S$, and every $1 \leq i, j \leq n$, there are at most $P(k)$ tuples $(i_0, \ldots, i_k) \in \{1, \ldots, n\}^{k+1}$ which satisfy the conditions $i_0 = i$, $i_k = j$, and $p_1[i_0, i_1] \wedge \cdots \wedge p_k[i_{k-1}, i_k] = 1$.

Let $p \in S$ and $1 \leq i, j \leq n$ satisfying $p[i, j] = 1$. We call (i, j) *unambiguous in* p if for every $r, s \in S$ satisfying $p = rs$, there is exactly one $1 \leq k \leq n$ such that $r[i, k] \wedge s[k, j] = 1$.

Assume that (i, j) is unambiguous in p. Let $k \geq 1$, $p_1, \ldots, p_k \in S$ satisfying $p = p_1 \cdots p_k$. There are unique $i = i_0, \ldots, i_k = j$ such that $p_1[i_0, i_1] \wedge \cdots \wedge p_k[i_{k-1}, i_k] = 1$. For every $1 \leq \ell \leq k$, the pair $(i_{\ell-1}, i_\ell)$ is unambiguous in p_ℓ.

Lemma 9. *Let $S \subseteq \mathbb{B}^{n \times n}$ be a polynomially ambiguous subsemigroup and $p \in S$.*

1. *For every i, j satisfying $p[i, i] = p[i, j] = p[j, i] = p[j, j] = 1$, we have $i = j$.*
2. *For every i satisfying $p[i, i] = 1$, the pair (i, i) is unambiguous in p.*

By Lemma 9(1), $\leq_e$ is antisymmetric.

Lemma 10. *Let S be a polynomially ambiguous subsemigroup of $\mathbb{B}^{n\times n}$. Let $C \subseteq \{1,\dots,n\}$ and let $e \in \mathsf{E}(S)$, and assume that e is stable on C. For every $i \in C$ which is minimal for $\leq_e$ in C, we have $e[i,i] = 1$.*

Proof. Let $i \in C$ be minimal. Since, $Ce = C$, we have $(Ce)[i] = 1$, i.e., there is a $j \in C$ such that $e[j,i] = 1$. Thus, $j \leq_e i$, and since i is minimal, we have $j = i$.

Lemma 11. *For every trim, polynomially ambiguous WFA $\mathcal{A} = [Q,\theta,\lambda,\varrho]$, the subsemigroup $\alpha(\theta(\Sigma^*)) \subseteq \mathbb{B}^{Q\times Q}$ is polynomially ambiguous.*

4.2 The Proofs of (2)⇒(3) and (3)⇒(1) in Theorem 4

Let $\mathcal{A} = [Q,\theta,\lambda,\varrho]$ be a polynomially ambiguous WFA for the rest of this section. Let $n := |Q|$ and assume $Q = \{1,\dots,n\}$. Let $T := \theta(\Sigma^*) \subseteq \mathbb{Z}^{n\times n}$ and $S := \alpha(\theta(\Sigma^*)) = \alpha(T) \subseteq \mathbb{B}^{n\times n}$. By Lemma 11, S is polynomially ambiguous.

Let $C \subseteq Q$ and $A \in \mathbb{Z}^{n\times n}$. By an abuse of notation, we define a product $CA \in \mathbb{Z}^n$, by setting for every $1 \leq i \leq n$, $(CA)[i] = \min_{\ell\in C} A[\ell,i]$.

In Section 3.2, we already defined $\min(B) := \min_{1\leq i\leq n} B[i]$ for $B \in \mathbb{Z}^n$. For every $B \in \mathbb{Z}^n \setminus \{(\infty,\dots,\infty)\}$, let

1. $\max(B) := \max_{1\leq i\leq n,\, B[i]\neq\infty} B[i]$,
2. $\mathsf{span}(B) := \max(B) - \min(B)$, and $\mathsf{span}\big((\infty,\dots,\infty)\big) := 0$.

Lemma 12. *The following assertions are equivalent.*

1. *The set $Q' = \{\mathsf{nf}(\lambda\theta(w)) \mid w \in \Sigma^*\}$ is finite.*
2. *There is some $K \in \mathbb{N}$ such that $\mathsf{span}(\lambda\theta(w)) \leq K$ for every $w \in \Sigma^*$.*

Proof. We get (1)⇒(2) by observing $\mathsf{span}(\mathsf{nf}(B)) = \mathsf{span}(B)$ for every $B \in \mathbb{Z}^n$. For (2)⇒(1), one shows $Q' \subseteq \{0,\dots,K,\infty\}^n$.

We have (1) in Lemma 12 iff Mohri's algorithm terminates on $\mathcal{A}$ (Section 3.2).

Proof ((2)⇒(3) in Theorem 4). By contradiction, we assume that (3) is false. Hence, there are $C \in \mathsf{Clones}(\mathcal{A})$ and $w \in \Sigma^*$ such that

(a) $e := \alpha(\theta(w)) \in \mathsf{E}(\mathbb{B}^{n\times n})$, C is stable on e, and
(b) there is some $p \in C$ such that $\theta(v)[p,p] \neq \infty$, and every state $p' \in C$ satisfying $\theta(w)[p',p] \neq \infty$ does not have a minimal cycle in C and $\theta(w)$.

Let $p' \in C$ such that p' is minimal for $\leq_e$ and $p' \leq_e p$. By Lemma 10, we have $e[p',p'] = 1$ and $\theta(w)[p',p'] \neq \infty$. Let $q \in C$, such that q has a minimal cycle at C and $\theta(w)$. We have $\theta(w)[q,q] \neq \infty$. By (b), we have $\theta(w)[p',p'] > \theta(w)[q,q]$.

Let $v \in \Sigma^*$ such that $C = \alpha(\lambda\theta(v))$. Let $k \geq 1$. We examine $\lambda\theta(vw^k)$.

At first, we consider $\lambda\theta(vw^k)[p']$. Since, (p',p') is unambiguous in e (by Lemma 9(2)), and since, p' is minimal in C, we can show

$$\lambda\theta(vw^k)[p'] \;=\; \big(\lambda\theta(v)\big)[p'] + k\cdot\big(\theta(w)[p',p']\big).$$

On the other hand, we have $\big(\lambda\theta(vw^k)\big)[q] \;\leq\; \big(\lambda\theta(v)\big)[q] + k\cdot\big(\theta(w)[q,q]\big) \;\neq\; \infty$. From $\theta(w)[p',p'] > \theta(w)[q,q]$, $\mathsf{span}(\lambda\theta(vw^k))$ tends to infinity for increasing k. By arguing as in Lemma 12, Mohri's algorithm does not terminate on $(vw^k)_{k\geq 1}$.

Lemma 13. *For $B \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{n \times n}$, we have* $\mathsf{span}(BA) \leq \mathsf{span}(B) + \mathsf{span}(\alpha(B)A)$.

The notion of the side entry bound is, beside the clones property, the most important notion of the paper. Let $C \in \mathsf{Clones}(\mathcal{A})$ and $A \in T$. We denote the *side entry bound* of C and A by $\mathsf{seb}(C, A)$ and define it as the least integer which satisfies $\mathsf{seb}(C, A) \geq \mathsf{span}(CA)$ and the following condition:

For every $i \in C$, $1 \leq j \leq n$ such that (i, j) is unambiguous in $\alpha(A)$, we have
if there is some $i' \in C \setminus \{i\}$ such that $A[i', j] \neq \infty$,
then there is some $\hat{i} \in C \setminus \{i\}$ such that $A[\hat{i}, j] \leq \min(CA) + \mathsf{seb}(C, A)$.

Lemma 14. *Let $A_1, A_2 \in T$, $C_1 \in \mathsf{Clones}(\mathcal{A})$ and $C_2 := \alpha(C_1 A_1) \in \mathsf{Clones}(\mathcal{A})$.*

1. *If $C_1 A_1 A_2 \neq (\infty, \ldots, \infty)$, then* $\min(C_1 A_1 A_2) \geq \min(C_1 A_1) + \min(C_2 A_2)$.
2. $\mathsf{span}(C_1 A_1 A_2) \leq \mathsf{span}(C_1 A_1) + \mathsf{span}(C_2 A_2)$
3. $\mathsf{seb}(C_1, A_1 A_2) \leq \mathsf{seb}(C_1, A_1) + \mathsf{seb}(C_2, A_2)$

Proof. (1) By $C_1 A_1 A_2 \neq (\infty, \ldots, \infty)$, there are $i \in C_1$, $j \in C_2$, $1 \leq \ell \leq n$ such that $\infty \neq \min(C_1 A_1 A_2) = C_1 A_1 A_2[\ell] = A_1[i, j] + A_2[j, \ell]$. Thus, $A_1[i, j] \neq \infty \neq A_2[j, \ell]$, i.e., $C_1 A_1 \neq (\infty, \ldots, \infty) \neq C_2 A_2$. We have $A_1[i, j] \geq C_1 A_1[j] \geq \min(C_1 A_1)$, $A_2[j, \ell] \geq C_2 A_2[\ell] \geq \min(C_2 A_2)$, and (1) follows.

(2) We use Lemma 13 for $B := C_1 A_1$ and $A := A_2$.

(3) Let $x := \mathsf{seb}(C_1, A_1) + \mathsf{seb}(C_2, A_2)$. We show $\mathsf{seb}(C_1, A_1 A_2) \leq x$. Above, $\mathsf{seb}(C_1, A_1 A_2)$ was defined as the least number which satisfies two conditions. We show that x satisfies these two conditions (3a, 3b), and thus, $\mathsf{seb}(C_1, A_1 A_2) \leq x$.

(3a) $x \geq \mathsf{span}(C_1 A_1 A_2)$
(3b) Let $i \in C_1$ and $1 \leq j \leq n$ such that (i, j) is unambiguous in $\alpha(A_1 A_2)$. We have to show that if there is some $i' \in C_1 \setminus \{i\}$ such that $A_1 A_2[i', j] \neq \infty$, then there is some $\hat{i} \in C_1 \setminus \{i\}$ such that $A_1 A_2[\hat{i}, j] \leq \min(C_1, A_1 A_2) + x$.

Since $\mathsf{seb}(C_1, A_1) \geq \mathsf{span}(C_1 A_1)$, $\mathsf{seb}(C_2, A_2) \geq \mathsf{span}(C_2 A_2)$, we have by (2) $x \geq \mathsf{span}(C_1 A_1) + \mathsf{span}(C_2 A_2) \geq \mathsf{span}(C_1 A_1 A_2)$ which proves (3a).

For (3b), let $i \in C_1$, $1 \leq j \leq n$ such that (i, j) is unambiguous in $\alpha(A_1 A_2)$. Let $1 \leq \ell \leq n$ such that $A_1 A_2[i, j] = A_1[i, \ell] + A_2[\ell, j]$. Since, (i, j) is unambiguous in $\alpha(A_1 A_2)$, (i, ℓ) resp. (ℓ, j) are unambiguous in $\alpha(A_1)$ resp. $\alpha(A_2)$. Let $i' \in C_1 \setminus \{i\}$ such that $A_1 A_2[i', j] \neq \infty$. If i' does not exist, then we are done.
Case 1: For every $\ell' \in C_2 \setminus \{\ell\}$, we have $A_2[\ell', j] = \infty$.
We have $C_2 A_2[j] = A_2[\ell, j]$, and hence $A_2[\ell, j] \leq \min(C_2 A_2) + \mathsf{span}(C_2 A_2)$.

We have $A_1[i', \ell] \neq \infty$. By the definition of $\mathsf{seb}(C_1, A_1)$, there is some $\hat{i} \in C_1 \setminus \{i\}$ such that $A_1[\hat{i}, \ell] \leq \min(C_1 A_1) + \mathsf{seb}(C_1, A_1)$. To sum up,

$$A_1 A_2[\hat{i}, j] \leq A_1[\hat{i}, \ell] + A_2[\ell, j] \leq \min(C_1 A_1) + \mathsf{seb}(C_1, A_1) +$$

$$+ \min(C_2 A_2) + \mathsf{span}(C_2 A_2) \leq \min(C_1 A_1 A_2) + \mathsf{seb}(C_1 A_1) + \mathsf{seb}(C_2, A_2).$$

Case 2: There is some $\ell' \in C_2 \setminus \{\ell\}$ such that $A_2[\ell', j] \neq \infty$.
By the definition of $\mathsf{seb}(C_2, A_2)$, there is some $\hat{\ell} \in C_2 \setminus \{\ell\}$ such that $A_2[\hat{\ell}, j] \leq \min(C_2 A_2) + \mathsf{seb}(C_2, A_2)$. By the definition of $\mathsf{span}(C_2 A_2)$, there is some $\hat{i} \in C_1$

such that $A_1[\hat{i},\hat{\ell}] \leq \min(C_1A_1) + \mathsf{span}(C_1A_1)$. We have $A_1A_2[\hat{i},j] \leq A_1[\hat{i},\hat{\ell}] + A_2[\hat{\ell},j]$ and proceed as in case 1. It remains to show $\hat{i} \neq i$. We have $\alpha(A_1)[\hat{i},\hat{\ell}] \wedge \alpha(A_2)[\hat{\ell},j] = 1$ and $\alpha(A_1)[i,\ell] \wedge \alpha(A_2)[\ell,j] = 1$. Since, $\hat{\ell} \neq \ell$ and (i,j) is unambiguous in $\alpha(A_1A_2)$, we have $\hat{i} \neq i$.

Lemma 15. *Assume that $\mathcal{A}$ satisfies the clones property. Let $k \geq 1$, $A_1,\ldots,A_k \in T$ such that $\alpha(A_1) = \cdots = \alpha(A_k) \in \mathsf{E}(S)$. Let $C \in \mathsf{Clones}(\mathcal{A})$ such that $\alpha(A_1)$ is stable on C.*

1. $\mathsf{span}(CA_1\cdots A_k) \leq 2(n-1)\max_{1\leq\ell\leq k}\mathsf{seb}(C,A_\ell)$
2. $\mathsf{seb}(C,A_1\cdots A_k) \leq 2n\max_{1\leq\ell\leq k}\mathsf{seb}(C,A_\ell)$

The bound on $\mathsf{span}(CA_1\cdots A_k)$ in Lemma 15(1) depends on $\mathsf{seb}(C,A_\ell)$ for $1 \leq \ell \leq k$. An example in [12] shows that is not possible to show an upper bound on $\mathsf{span}(CA_1\cdots A_k)$ which is independent depends on $\mathsf{seb}(C,A_\ell)$ for $1 \leq \ell \leq k$.

Proof (Proof of Lemma 15). Denote $e := \alpha(A_1)$ and $A = A_1\ldots A_k$. We assume $C \neq \emptyset$ and $k \geq 2$ since otherwise, the proof is easy.

Let $p \in C$ be minimal for $\leq_e$. By Lemma 10, we have $e[p,p] = 1$, and hence, $A_\ell[p,p] \neq \infty$ for every $1 \leq \ell \leq k$. Since, p is minimal, we have for every $1 \leq \ell \leq k$,

$$(CA_1\cdots A_\ell)[p] = (A_1\cdots A_\ell)[p,p] = \sum_{1\leq\ell'\leq\ell} A_{\ell'}[p,p]. \tag{4.1}$$

By the clones property, we have for every $1 \leq \ell \leq k$, $q \in C$, $A_\ell[p,p] \leq A_\ell[q,q]$. Let $\mathsf{mx}_{\mathsf{span}} := \max_{1\leq\ell\leq k}\mathsf{span}(CA_\ell)$ and $\mathsf{mx}_{\mathsf{seb}} := \max_{1\leq\ell\leq k}\mathsf{seb}(C,A_\ell)$. By definition, $\mathsf{mx}_{\mathsf{span}} \leq \mathsf{mx}_{\mathsf{seb}}$. Next, we show the claims (C1) and (C2). Finally, we derive (1) of the lemma from (C1) and (C2). The proof of (2) is similar [12].

(C1) For every $j \in C$, we have $CA[j] \geq CA[p] - (n-1)\mathsf{mx}_{\mathsf{span}}$.
(C2) For every $1 \leq \ell \leq k$, $j \in C$, we have $(CA_1\cdots A_\ell)[j] \leq (CA_1\cdots A_\ell)[p] + (n-1)\mathsf{mx}_{\mathsf{seb}}$.

We show (C1). Let $j \in C$. Let $1 \leq i_0,\ldots,i_k \leq n$, $i_k = j$ such that for every $1 \leq \ell \leq k$, we have $A_\ell[i_{\ell-1},i_\ell] \neq \infty$. For every $1 \leq \ell \leq k$, we have $e[i_{\ell-1},i_\ell] = 1$, i.e., $i_0 \leq_e i_1 \leq_e \ldots \leq_e i_k$. Since, e is stable on C, we have $i_0,\ldots,i_k \in C$.

Let $1 \leq \ell \leq k$ such that $i_{\ell-1} \neq i_\ell$. Since p is minimal, we have $CA_\ell[p] = A_\ell[p,p]$. We have $A_\ell[i_{\ell-1},i_\ell] \geq CA_\ell[p] - \mathsf{span}(CA_\ell) = A[p,p] - \mathsf{span}(CA_\ell)$.

Let $1 \leq \ell \leq k$ such that $i_{\ell-1} = i_\ell$. As seen above, we have $A_\ell[i_{\ell-1},i_\ell] \geq A_\ell[p,p]$ for every $1 \leq \ell \leq k$. From these bounds on $A_\ell[i_{\ell-1},i_\ell]$, we obtain

$$\sum_{1\leq\ell\leq k} A_\ell[i_{\ell-1},i_\ell] \geq \sum_{1\leq\ell\leq k} A_\ell[p,p] - \sum_{1\leq\ell\leq k,\, i_{\ell-1}\neq i_\ell} \mathsf{span}(CA_\ell)$$

Since, S is polynomially ambiguous, $\leq_e$ is antisymmetric and transitive, i.e., there are at most $n-1$ integers $1 \leq \ell \leq k$ such that $i_{\ell-1} \neq i_\ell$. Hence, we have

$$\sum_{1\leq\ell\leq k} A_\ell[i_{\ell-1},i_\ell] \geq A[p,p] - (n-1)\mathsf{mx}_{\mathsf{span}},$$

and thus, $CA[j] \geq CA[p] - (n-1)\mathsf{mx}_{\mathsf{span}}$.

We show (C2). Let $j \in C$. We define $\mathsf{ind}(j) := \big|\{i \in C \mid i \leq_e j,\, i \neq j\}\big|$. We have $0 \leq \mathsf{ind}(j) < n$. For every $i \leq_e j$ satisfying $i \neq j$, we have $\mathsf{ind}(i) < \mathsf{ind}(j)$. For (C2), we show by an induction on $\leq_e$ for every $j \in C$, $1 \leq \ell \leq k$:

$$(CA_1 \cdots A_\ell)[j] \;\leq\; (CA_1 \cdots A_\ell)[p] + \mathsf{ind}(j)\mathsf{mx}_{\mathsf{seb}}. \tag{4.2}$$

Let $j \in C$ be minimal for $\leq_e$, i.e., $\mathsf{ind}(j) = 0$. Since, $\mathcal{A}$ satisfies the clones property, we have for every $1 \leq \ell' \leq k$, $A_{\ell'}[j,j] = A_{\ell'}[p,p]$. We have for every $1 \leq \ell \leq k$, $(CA_1 \cdots A_\ell)[j] = (A_1 \cdots A_\ell)[j,j] = \sum_{1 \leq \ell' \leq \ell} A_{\ell'}[j,j]$. In combination with (4.1), we obtain $(CA)[j] = (CA)[p]$ which proves (4.2) for j.

Let $j \in C$ and assume by induction, that (4.2) holds for every $1 \leq \ell \leq k$, $i \leq_e j$, $i \neq j$. Assume that j is not minimal for $\leq_e$ in C. We show that there exists some $i \in C$ such that $i \neq j$, $i \leq_e j$, and $A_\ell[i,j] \leq A_\ell[p,p] + \mathsf{seb}(C, A_\ell)$.
Case 1. $e[j,j] = 0$. We have $A_\ell[j,j] = \infty$. Let $i \in C$ such that $A_\ell[i,j] = CA_\ell[j]$. We have $CA_\ell[j] \leq CA_\ell[p] + \mathsf{span}(CA_\ell)$, i.e., $A_\ell[i,j] \leq A_\ell[p,p] + \mathsf{span}(CA_\ell) \leq A_\ell[p,p] + \mathsf{seb}(C, A_\ell)$. Since $A_\ell[i,j] \neq \infty$, we have $e[i,j] = 1$, i.e., $i \neq j$ and $i \leq_e j$.
Case 2. $e[j,j] = 1$. By Lemma 9, (j,j) is unambiguous in e. Since, j is not minimal for $\leq_e$ in C, there is some i' such that $e[i',j] = 1$, i.e., $A_\ell[i',j] \neq \infty$. By the definition of $\mathsf{seb}(C, A_\ell)$, there is some $i \in C \setminus \{j\}$ such that $A_\ell[i,j] \leq \min(CA_\ell) + \mathsf{seb}(C, A_\ell)$, i.e., $A_\ell[i,j] \leq A_\ell[p,p] + \mathsf{seb}(C, A_\ell)$. Obviously, $i \neq j$ and since, $A_\ell[i,j] \neq \infty$, we have $e[i,j] = 1$, i.e., $i \leq_e j$, which completes case 2.

We have (4.2) for $\ell = 1$ by $(CA_1)[j] \leq A_1[i,j] \leq A_1[p,p] + \mathsf{seb}(C, A_1) \leq (CA_1)[p] + \mathsf{seb}(C, A_1) \leq (CA_1)[p] + \mathsf{ind}(j)\mathsf{mx}_{\mathsf{seb}}$. We show (4.2) for $2 \leq \ell \leq k$. By the induction on $(CA_1 \cdots A_{\ell-1})[i]$ and the bound on $A_\ell[i,j]$, we get

$$\begin{aligned}(CA_1 \cdots A_\ell)[j] \quad &\leq \quad (CA_1 \cdots A_{\ell-1})[i] + A_\ell[i,j] \quad \leq \\ \leq \quad &(CA_1 \cdots A_{\ell-1})[p] + \mathsf{ind}(i)\mathsf{mx}_{\mathsf{seb}} + A_\ell[p,p] + \mathsf{seb}(C, A_\ell) \quad \leq \\ \leq \quad &(CA_1 \cdots A_\ell)[p] + \mathsf{ind}(j)\mathsf{mx}_{\mathsf{seb}} \quad \text{which proves (4.2) for } j.\end{aligned}$$

By (C1)(C2) for $\ell = k$, we get for $j \in C$, $CA[p] - (n-1)\mathsf{mx}_{\mathsf{seb}} \leq CA[j] \leq CA[p] + (n-1)\mathsf{mx}_{\mathsf{seb}}$, i.e., $\mathsf{span}(CA) \leq (n-1)(\mathsf{mx}_{\mathsf{span}} + \mathsf{mx}_{\mathsf{seb}})$ which proves (1).

Theorem 16 (factorization forest theorem [20,3]). *Let S be a finite semigroup and $h : \Sigma^* \to S$ be a homomorphism. There is a mapping $d : \Sigma^* \to \{1, \ldots, 7|S|\}$ such that every $w \in \Sigma^*$ satisfies the following two conditions:*

1. *if $d(w) = 1$, then $|w| \leq 1$, and*
2. *if $d(w) \geq 2$, then there are some $k \geq 2$, $w_1, \ldots, w_k \in \Sigma^+$ such that*
   (a) $w_1 \ldots w_k = w$, for every $1 \leq \ell \leq k$, $d(w_\ell) < d(w)$, and
   (b) if $k \geq 3$, then $h(w_1) = \cdots = h(w_k) = h(w) \in \mathsf{E}(S)$.

Proof (sketch of (3)⇒(1) in Theorem 4). Let $d : \Sigma^* \to \{1, \ldots, 7|S|\}$ be as in Theorem 16 for $\alpha \circ \theta : \Sigma^* \to S$. Let $\mathsf{mx}_{\mathsf{seb}} := \max_{C \in \mathsf{Clones}(\mathcal{A}),\, a \in \Sigma} \mathsf{seb}(C, \theta(a))$.

We show the following claim by an induction on $d(w)$:

(C) For every $w \in \Sigma^*, C \in \mathsf{Clones}(\mathcal{A})$, we have $\mathsf{seb}(C, \theta(w)) \leq (2n+1)^{d(w)-1} \cdot \mathsf{mx}_{\mathsf{seb}}$.

Claim (C) is sufficient for the termination of MOHRI's algorithm on $\mathcal{A}$.

Let $w \in \Sigma^*$. If $d(w) = 1$ then $w \in \Sigma$ and (C) follows from the def. of $\mathsf{mx}_{\mathsf{seb}}$.

Assume $d(w) \geq 2$. Let $k \geq 2$ and $w_1, \ldots, w_k \in \Sigma^+$ as in Theorem 16(2). For $1 \leq \ell \leq k$, we have $d(w_\ell) < d(w)$, i.e., (C) is true for w_ℓ. We show (C) for w. If $k = 2$, then (C) follows from Lemma 14(3) and the inductive hypothesis. Assume $k \geq 3$. Let $C \in \mathsf{Clones}(\mathcal{A})$ and $C' := C\alpha(\theta(w_1))$. The clone C' is stable on $\alpha(\theta(w_1))$. We can show (C) by applying Lemma 15(2) on C' and $\theta(w_2), \ldots, \theta(w_k)$ and applying Lemma 14(3) on C, $\theta(w_1)$ and C', $\theta(w_2 \ldots w_k)$.

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