Generation of discrete planes

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Abstract

We consider the problem of generation of discrete planes using generalized substitutions. We give sufficient conditions to be sure to generate all of a discrete plane by a sequence of substitutions; these conditions, however, are not easy to check, even on simple examples.

One can build approximations of discrete planes in several ways, namely as stepped surfaces (unions of faces), as sets of vertices, or as two-dimensional sequences on a three-letter alphabet; these codings are equivalent in some cases, as we recall below. Recent progresses have given a way to act on these approximations by generalized substitutions; but an open question is to know whether we can, in this way, generate all (or an arbitrarily large neighborhood of the origin) of the discrete plane.

In this short communication, we give a sufficient condition to generate an arbitrarily large neighborhood of the origin; unfortunately, this condition seems, at the moment, to be difficult to check on explicit examples, even in simple cases.

1 Generation of discrete planes: definitions and known results

1.1 Discrete planes

Let $\mathcal{P}_{(a,b,c)} \subset \mathbb{R}^3$ be a plane with equation $ax + by + cz = 0$. We suppose that $a, b, c > 0$. We want to approximate the plane $\mathcal{P}$ by a *stepped surface*, defined as a union of faces of integral cubes. We thus introduce the *discrete plane* approximation $\mathfrak{P}$ of the plane $\mathcal{P}_{(a,b,c)}$ as the upper boundary of the union of all unit cubes with integral vertices that intersect this plane. This construction is inspired by the cut-and-project formalism in quasicrystals.

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denote the canonical basis of $\mathbb{R}^3$. We call *integral cube* any translate of the fundamental cube with integral vertices, that is, any set $(\mathbf{p}, \mathbf{q}, \mathbf{r}) + \mathcal{C}$ where $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathbb{Z}^3$ and $\mathcal{C}$ stands for the fundamental unit cube:

$$\mathcal{C} = \{\lambda \mathbf{e_1} + \mu \mathbf{e_2} + \nu \mathbf{e_3},\ (\lambda, \mu, \nu) \in [0, 1]^2\}.$$

Definition 1. [1, 4] Let $\mathcal{S}$ be the set of integral cubes that intersect the lower half-space $ax + by + cz < 0$.

The *discrete plane* or *stepped surface* associated with $\mathcal{P}_{(a,b,c)}$ is the boundary of the set $\mathcal{S}$. This discrete plane is denoted by $\mathfrak{P}_{(a,b,c)}$.

A *vertex* of the discrete plane $\mathfrak{P}_{(a,b,c)}$ is an integral point that belongs to the discrete plane. Let $\mathcal{V}_{(a,b,c)}$ stand for the set of vertices of $\mathfrak{P}_{(a,b,c)}$.

Let F_1, F_2, and F_3 be the three following basic faces for the discrete plane:

$$\begin{array}{rcl} F_1 & = & \{\lambda \mathbf{e}_2 + \mu \mathbf{e}_3,\ (\lambda, \mu) \in [0, 1[^2\} \\ F_2 & = & \{\lambda \mathbf{e}_1 + \mu \mathbf{e}_3,\ (\lambda, \mu) \in [0, 1[^2\} \\ F_3 & = & \{\lambda \mathbf{e}_1 + \mu \mathbf{e}_2,\ (\lambda, \mu) \in [0, 1[^2\}. \end{array}$$

We call *face of type i with basic vertex* (p, q, r) the set $(p, q, r) + F_i$. The point (p, q, r) is called the *basic vertex*, or lower vertex, of the face $(p, q, r) + F_i$. This definition has the advantage of giving a symmetric role to the three coordinates, and is more convenient for computation. It has however a major inconvenience: some vertices of the stepped surface are not the basic vertex of a tile, and some are the basic vertices of several tiles.

We can remediate to this problem by a small change of notations; we denote by E_1, E_2, and E_3 the three following sets:

$$\begin{array}{rcl} E_1 & = & \{\lambda \mathbf{e}_2 + \mu \mathbf{e}_3,\ (\lambda, \mu) \in [0, 1[^2\} \\ E_2 & = & \{-\lambda \mathbf{e}_1 + \mu \mathbf{e}_3,\ (\lambda, \mu) \in [0, 1[^2\} \\ E_3 & = & \{-\lambda \mathbf{e}_1 - \mu \mathbf{e}_2,\ (\lambda, \mu) \in [0, 1[^2\}. \end{array}$$

We call *face of type i pointed on* (p, q, r) or shortly *pointed face* the set $(p, q, r) + E_i$. The point (p, q, r) is called the *distinguished vertex* of the face $(p, q, r) + E_i$.

Remark 1. This is just a change of coordinates for the stepped surface: we clearly have $F_1 = E_1$, $F_2 = E_2 + \mathbf{e}_1$, $F_3 = E_3 + \mathbf{e}_1 + \mathbf{e}_2$. The important change is in the vertex associated with a face: while the basic vertex is always the lowest vertex (with respect to the height function $x + y + z$), it is not true for the distinguished vertex, and this has a nice consequence, as shown in the next lemma.

Lemma 1.1. *The map that sends any face of the stepped surface to its distinguished vertex is a bijection from the set of faces of the stepped surface to the set of vertices of this stepped surface.*

Proof. It suffices to remark that a vertex is distinguished in a face E if, when we consider the orthogonal projection π onto the diagonal plane $x + y + z = 0$, then the vector $\pi(e_3 - e_1)$, starting from the distinguished vertex, points inside the projection of the face. Is is then clear that a face has exactly one distinguished vertex, and that each vertex is the distinguished vertex of exactly one tile. $\square$

The distinguished vertices in $\mathcal{V}_{(a,b,c)}$ are easy to compute:

Proposition 1.2. *A point $(p,q,r) \in \mathbb{Z}^3$ is the distinguished vertex of a face of type 1 (resp. 2 or 3) in the discrete plane $\mathfrak{P}_{(a,b,c)}$ if and only if $ap+bq+cr \in [0,a)$ (resp. $[a,a+b)$ or $[a+b,a+b+c)$). This is equivalent with $(p,q,r)+E_i$ to be included in $\mathfrak{P}_{(a,b,c)}$. If so, (p,q,r) belongs to no other face in the discrete plane.*

Let $\mathfrak{P}^*_{(a,b,c)}$ denote the set of finite pointed patterns on the discrete plane, that is, the set of finite disjoint unions of faces in the discrete plane. A consequence of the previous proposition implies that E_1, $(1,0,0)+E_2$ and $(1,1,0)+E_3$ all belong to $\mathfrak{P}_{(a,b,c)}$ (hese three faces are the three faces F_1, F_2, F_3 at the origin). We denote by $\mathcal{U}$ the union of these three faces; this is the largest pattern that belongs to all the planes $\mathcal{P}_{(a,b,c)}$ independently of their direction:

$$\forall (a,b,c), \qquad \mathcal{U} = E_1 \cup (1,0,0)+E_2 \cup (1,1,0)+E_3 \subset \mathfrak{P}_{(a,b,c)}.$$

1.2 Two-dimensional words

We can code a discrete plane by a two-dimensional sequence in the following way. We recall that π stands for the projection onto the diagonal plane $x+y+z=0$ along the direction $(1,1,1)$. This projection sends the lattice $\mathbb{Z}^3$ to a lattice Γ in the diagonal plane. A simple computation gives: $\pi(p,q,r) = (p-r)\pi(\mathbf{e}_1) + (q-r)\pi(\mathbf{e}_2)$, hence $\Gamma = \mathbb{Z}\,\pi(\mathbf{e}_1) + \mathbb{Z}\,\pi(\mathbf{e}_2)$.
One checks that the restriction of π to $\mathcal{V}_{(a,b,c)}$ is a bijection on its image Γ. This allows us to define a sequence, by associating with any point in Γ the type of the face having its preimage as a distinguished vertex, as made precise in the theorem.

Theorem 1.3. *Let $(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{\mathbf{2}}$ and $\mathbf{g} = m\pi(\mathbf{e}_1)+n\pi(\mathbf{e}_2)$ in the lattice Γ. There exists a unique integer $U(\mathbf{m},\mathbf{n}) \in \{1,2,3\}$ such that $\pi^{-1}(\mathbf{g})$ is the distinguished vertex of a face of type $U(\mathbf{m},\mathbf{n})$ in the discrete plane $\mathfrak{P}_{(a,b,c)}$:*

$$\begin{aligned} U_{(a,b,c)}(\mathbf{m},\mathbf{n}) &= 1 \quad &\textit{if} \quad &(am+bn) \ \textit{mod}\ (a+b+c) \in [0,a),\\ U_{(a,b,c)}(\mathbf{m},\mathbf{n}) &= 2 \quad &\textit{if} \quad &(am+bn) \ \textit{mod}\ (a+b+c) \in [a,a+b),\\ U_{(a,b,c)}(\mathbf{m},\mathbf{n}) &= 3 \quad &\textit{if} \quad &(am+bn) \ \textit{mod}\ (a+b+c) \in [a+b,a+b+c). \end{aligned}$$

The sequence $(U_{(a,b,c)}(\mathbf{m},\mathbf{n}))_{\mathbb{Z}^{\mathbf{2}}}$ is called the the two-dimensional coding *associated with the plane $ax+by+c=0$.*

It is slightly misleading to think of Γ as $\mathbb{Z}^2$; one should think of this lattice as a centered hexagonal lattice, like the lattice $\mathbb{Z}(j) \in \mathbb{C}$, with $j = \frac{-1+i\sqrt{3}}{2}$. Let us note that a sufficient and necessary condition for a two-dimensional pattern to be a factor of the two-dimensional coding of a discrete plane is given in [7].

1.3 Generalized substitutions

We now define generalized substitutions, that is, maps that act on the faces in a stepped surface.

1.3.1 One-dimensional iterated morphisms

Let $\mathcal{A}$ be the finite alphabet $\{1, 2, 3\}$ and $\mathcal{A}^*$ the set of finite words defined over $\mathcal{A}$. The empty word is denoted by ε. A one-dimensional *iterated morphism* σ is an endomorphism of the free-monoid $\mathcal{A}^*$ such that the image of a letter of $\mathcal{A}$ is never empty; we also require that for at least one letter a, we have $|\sigma^n(a)| \to +\infty$, where $|w|$ stands for the length of the word w. It extends in a natural way to infinite or biinfinite sequences in $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$.

1.3.2 Abelianization

Let $\mathbf{l} : \mathcal{A}^* \mapsto \mathbb{N}^3$ be the natural homomorphism obtained by abelianization of the free monoid: if $|W|_a$ denotes the number of occurrences of the letter $a \in \mathcal{A}$ in a finite word W, then we have $\mathbf{l}(W) = (|W|_1, |W|_2, |W|_3) \in \mathbb{N}^3$.
With each one-dimensional iterated morphism σ on $\mathcal{A}$ is canonically associated its *incidence matrix* $\mathbf{M} = (m_{i,j})_{1\leq i,j\leq 3}$ defined by $m_{i,j} = |\sigma(j)|_i$ (where $|W|_i$ stands for the number of occurrences of the letter i in W), so that we have $\mathbf{l}(\sigma(W)) = \mathbf{M}\mathbf{l}(W)$ for every $W \in \mathcal{A}^*$.
An iterated morphism σ is *unimodular* if $\det \mathbf{M} = \pm 1$.

1.3.3 Generalized substitution

In [2, 3], for any unimodular iterated morphism, one defines the generalized substitution Σ_σ, acting on faces $(p, q, r) + F_i = \mathbf{x} + F_i$, by:

Definition 2. Let σ be a one-dimensional unimodular iterated morphism on three letters. We call *generalized substitution acting on faces* the following tranformation, denoted by Σ_σ, and defined on a face $\mathbf{x} + F_i$ by:

$$\Sigma_\sigma(\mathbf{x} + F_i) = \bigcup_{k\in\{1,2,3\}} \bigcup_{S,\,\sigma(k)=PiS} (\mathbf{M}^{-1}[\mathbf{x} + \mathbf{l}(S)]) + F_k. \tag{1.1}$$

If we want to use the notation E_i (distinguished vertex instead of lower vertex), which is necessary when we want to look at the action on two-dimensional words, the previous formula, by a short computation, becomes:

$$\Sigma_\sigma(\mathbf{x} + E_i) = \bigcup_{k\in\{1,2,3\}} \bigcup_{S,\,\sigma(k)=PiS} (\mathbf{M}^{-1}[\mathbf{x} - \mathbf{l}(S) - (\mathbf{e}_1 + \cdots + \mathbf{e}_i)] + (\mathbf{e}_1 + \cdots + \mathbf{e}_k)) + E_k. \tag{1.2}$$

Note that by construction, the composition of the generalized substitutions associated with σ_1 and σ_2 is the generalized substitution associated with to $\sigma_2 \circ \sigma_1$.
The generalized substitutions act on the discrete planes, by the following theorem (see [6]):

Theorem 1.4. *Any one-dimensional unimodular iterated morphism σ over a three-letter alphabet can be extended to a generalized substitution Σ_σ acting on faces of any discrete plane $\mathfrak{P}_{(a,b,c)}$. This generalized substitution maps any pattern (that is, a finite disjoint union of faces in the discrete plane) of $\mathfrak{P}_{(a,b,c)}$ on a pattern of the discrete plane $\mathfrak{P}_{(a_1,b_1,c_1)}$ where ${}^t(a, b, c) = {}^t\mathbf{M}_\Sigma\, {}^t(a_1, b_1, c_1)$.*
Furthermore the images of two distinct faces do not intersect, and $\mathcal{U} \subset \Sigma_\sigma(\mathcal{U})$.

The faces that occur in the image of a face of type i are associated with all the occurrences of the letter i in the images of the letters of $\{1, 2, 3\}$. The incidence matrix of Σ is hence the dual of that of σ:

$$\mathbf{M}_{\Sigma_\sigma} = {}^t\mathbf{M}_\sigma.$$

1.3.4 Iterated morphism of Pisot type

A morphism σ on three letters is *of Pisot type* if its eigenvalues satisfy $\alpha > 1 > |\lambda_1| \geq |\lambda_2| > 0$. In particular, the dominant eigenvalue α is a Pisot number. Furthermore, its incidence matrix $\mathbf{M}$ is primitive [5, 8], that is, it admits a power with strictly positive entries.
If the morphism is not of Pisot type, then, if we iterate a patch of a discrete plane, we do not generate a complete plane: the image patch extends in the direction of the second largest eigenvalue, and approximates a line, not a plane. This is why morphisms of the Pisot type are interesting: they are the ones for which we can expect to obtain the full plane by iterating on the patch $\mathcal{U}$. Indeed, the contracting discrete plane is invariant under the action of the substitution on faces. When we start with the patch $\mathcal{U}$ and iterate the substitution, we get larger and larger patches. It is not difficult to show:

Theorem 1.5. *Let σ be a one-dimensional unimodular iterated morphism of Pisot type, and let $\mathfrak{P}$ be the discrete plane approximating the contracting plane of M_σ. Then there exists a finite patch $\mathcal{V}$ in $\mathfrak{P}$ such that $\mathfrak{P} = \cup_{n\in\mathbb{N}}\Sigma_\sigma^n(\mathcal{V})$.*

However, the size of the patch $\mathcal{V}$ given by the proof of the theorem depends strongly on σ. We would like to give conditions under which the plane is obtained from the basic patch $\mathcal{U}$. This is not always the case, as shown by the example of the substitution $1 \mapsto 132323$, $2 \mapsto 23$, $3 \mapsto 323132323$, obtained from a special case of Jacobi-Perron algorithm; Figure 1 shows, in grey, a part of the plane generated by the patch $\mathcal{U}$, and in black, another part that contains a fixed point of the generalized substitution, and can never be attained by $\mathcal{U}$.
In the following section, we will give conditions that ensure that a substitution, or sequence of substitutions, generates the whole discrete plane starting from $\mathcal{U}$.

2 The ring lemma

2.1 Distance for a discrete plane

We want to define neighborhoods for a subset of the discrete plane; the easiest way is to define a distance. There are several equivalent ways to do this; we could use the distinguished vertices, project to the diagonal lattice and use any distance on this lattice. We choose a dfinition more closely linked to the geometry of the discrete plane.

Definition 3. Let $\mathfrak{P}_{(a,b,c)}$ be a discrete plane. We say that a finite sequence of faces $A_0, A_1, ..., A_n$ is a *path* of length n in $\mathfrak{P}_{(a,b,c)}$ if all the faces are contained in the discrete plane, and every pair of consecutive faces share a common edge. We say that this path *joins* A_0 and A_n. For any two faces A, B, there is a finite path that join them.

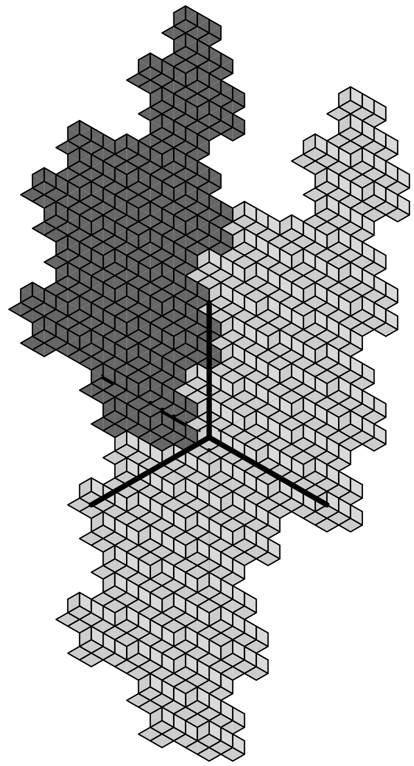

Figure 1: A patch of the discrete plane approximating the contracting plane for a particular Pisot substitution

Definition 4. The distance of two faces is the length of the shortest path that joins them. The ball of radius n around a face A in the discrete plane $\mathfrak{P}_{(a,b,c)}$, denoted by $\mathcal{B}_{(a,b,c)}(A,n)$, is the union of faces at distance at most n of A. The neighborhood of size n of a patch $\mathcal{W}$, also denoted by $\mathcal{B}_{(a,b,c)}(\mathcal{W},n)$, is the union of the balls of radius n around the faces in $\mathcal{W}$.

Remark 2. It is immediate that the neighborhood of size n of the neighborhood of size p of a patch $\mathcal{W}$ in $\mathfrak{P}_{(a,b,c)}$ is the neighborhood of size $n+p$ of $\mathcal{W}$.

We introduce two main properties that will be used as hypotheses in the ring lemma.

2.2 The surrounding hypothesis

We want the image of a neighborhood to be a neighborhood of the image:

Definition 5. A generalized substitution Σ_σ is said to satisfy the *surrounding property* if for every discrete plane, and every face $\mathbf{p}+E_i$ of this discrete plane, the image by Σ_σ of the ball of radius 1 around the face contains the neighborhood of size 1 of the image of the face.

Proposition 2.1. *If* Σ_σ *satisfies the* surrounding property *then the image of the neighborhood of size 1 of any patch* $\mathcal{W}$ *contained in any discrete plane contains a neighborhood of size 1 of the image of the set.*

Proof. Let B be a face contained in a neighborhood of size 1 of the image of $\mathcal{W}$. Suppose that B is not in the image of $\mathcal{W}$, otherwise there is nothing to prove. Then we can find a face $A \in \mathcal{W}$ such that B is in a neighborhood of size 1 of the image of A. By the surrounding property, B is in the image of a face A' which is at distance 1 from A, which proves the result. □

This property also works for neighborhood of arbitrary size:

Corollary 2.2. *If* Σ_σ *satisfies the* surrounding property *then the image of the neighborhood of size* n *of any patch* $\mathcal{W}$ *contained in a discrete plan contains a neighborhood of size* n *of the image of the set.*

Proof. Immediate by recurrence. □

2.3 The generation hypothesis

We introduce a second property meaning that the unit cube $\mathcal{U}$, which is always contained in its image, generates a neighborhood of itself under Σ_σ.

Definition 6. Let Σ_σ be a generalized substitution. Σ_σ satisfies the *generation hypothesis* if $\Sigma_\sigma(\mathcal{U})$ contains a neighborhood of size 1 of $\mathcal{U}$.

Remark 3. We know that $\mathcal{U} \subset \mathfrak{P}_{(\alpha,\beta,\gamma)}$ for all positive (a, b, c). Then $\Sigma_\sigma(\mathcal{U}) \subset \Sigma_\sigma(\mathfrak{P}_{(\alpha,\beta,\gamma)}) = \mathfrak{P}_{{}^tM_{\Sigma_\sigma}(a,b,c)}$. Hence, the neighborhood we get in the case of the generation hypothesis does not depend of the plane we consider: it is valid for all planes that can be obtained as image of a discrete plane by Σ_σ.

2.4 The ring lemma

Theorem 2.3 (Ring Lemma)**.** *Let* (Σ_{σ_n}) *be a set of generalized substitutions that all satisfy the generation hypothesis and the surrounding hypothesis. Then for every* $n \in \mathbb{N}$ *and every* (a, b, c) *positive which is the image of a positive vector by* $M_{\sigma_1} \dots M_{\sigma_n}$*, the composition* $\Sigma_{\sigma_1} \dots \Sigma_{\sigma_n}(\mathcal{U})$ *contains the neighborhood of size* n *of* $\mathcal{U}$ *in* $\mathfrak{P}_{(a,b,c)}$*:*

$$\mathcal{B}_{(a,b,c)}(\mathcal{U}, n) \subset \Sigma_{\sigma_1} \dots \Sigma_{\sigma_n}(\mathcal{U}). \tag{2.1}$$

Proof. This theorem is proved by induction.
If $n = 1$, then Eq. (2.1) reduces to the generation hypothesis on Σ_{σ_1}.
Suppose that the property is true for $n-1$, and let (a_1, b_1, c_1) be the coordinates of the preimage of the discrete plane (so that $(a, b, c) = {}^tM(a_1, b_1, c_1)$). Then we obtain that $\Sigma_{\sigma_2} \dots \Sigma_{\sigma_n}(\mathcal{U})$ is the neighborhood of size $n-1$ of $\mathcal{U}$ in $\mathfrak{P}_{(a_1,b_1,c_1)}$. But since $\Sigma_{\sigma_1}(\mathcal{U})$ contains a neighborhood of size 1 of $\mathcal{U}$, $\Sigma_{\sigma_1}\Sigma_{\sigma_2} \dots \Sigma_{\sigma_n}(\mathcal{U})$ contains a neighborhood of size $n-1$ of a neighborhood of size 1 of $\mathcal{U}$, by Corollary 2.2, hence a neighborhood of size n of $\mathcal{U}$, by Remark 2, which proves the result. □

This gives us a condition to generate completely the discrete plane:

Corollary 2.4. *Under the hypothesis of the ring lemma, we can generate the whole discrete plane $\mathfrak{P}_{(a,b,c)}$ by the sequence of generalized substitutions:*

$$\mathfrak{P}_{(a,b,c)} = \cup_{n \in \mathbb{N}} \Sigma_{\sigma_1} \ldots \Sigma_{\sigma_n}(\mathcal{U}).$$

3 Possible applications

Generalized continued fractions in dimension 2 give rise to sequences of substitutions (so-called S-adic systems), and approximations of discrete planes, so that it is reasonable to try to generate a plane in this way.
However, in the general case, the generating property does not hold, even if we consider powers of the maps, as shown in Figure 1. It seems reasonable in a first time to reduce our consideration to a system with better properties.

3.1 A special type of adic system

Let $\mathcal{T}_0$ be the set of points in $[0,1]^3$ that do not satisfy the triangular inequality:

$$\mathcal{T}_0 = \{(x,y,z) \in [0,1]^3,\ x+y<z \text{ or } x+z<y \text{ or } y+z<x\}.$$

We define three transformations on $\mathbb{R}^3$:

$$\begin{aligned} f^{(1)}(x,y,z) &= (x, y, z-x-y) \\ f^{(2)}(x,y,z) &= (x, y-x-z, z) \\ f^{(3)}(x,y,z) &= (x-y-z, y, z) \end{aligned}$$

and three associated matrices:

$$\mathbf{M}^{(1)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{M}^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{M}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Then the relation between a point (x,y,z) and its image $(x_1,y_1,z_1) = f(x,y,z)$ are related by the following relation:

$$(x_1,y_1,z_1) = f^{(i)}(x,y,z) \implies {}^t(x,y,z) = \mathbf{M}^{(i)t}(x_1,y_1,z_1)$$

The following extended continued fraction algorithm on $\mathcal{T}$ makes uses of functions $f^{(i)}$:

$$\begin{cases} & f^{(1)}(x,y,z) \quad \text{if } x+y<z \quad \text{(type 1)} \\ f : (x,y,z) \in \mathcal{T} \to & f^{(2)}(x,y,z) \quad \text{if } x+z<y \quad \text{(type 2)} \\ & f^{(3)}(x,y,z) \quad \text{if } y+z<x \quad \text{(type 3)} \end{cases}$$

Let $\mathcal{T}$ be the space stable by infinite iterations of f: $\mathcal{T} = \cap f^{-n}\mathcal{T}_0$.

Let $(a,b,c) \in \mathcal{T}$, and $(x_n, y_n, z_n) = f^n(a,b,c)$. If i_{k+1} denotes the type of transformation used to map (x_k, y_k, z_k) to $(x_{k+1}, y_{k+1}, z_{k+1})$ (that is, $(x_{k+1}, y_{k+1}, z_{k+1}) = f^{(i_{k+1})}(x_k, y_k, z_k)$), then the following relation holds:

$$^t(a,b,c) = \mathbf{M}^{(i_1)} \dots \mathbf{M}^{(i_n)t}(x_n, y_n, z_n).$$

Note that as soon as the three types 1, 2 and 3 appear infinitely often in the sequence i_k, the product of matrices $\mathbf{M}^{(i_1)} \dots \mathbf{M}^{(i_n)}$ is strictly contracting, so that $\mathbf{M}^{(i_1)} \dots \mathbf{M}^{(i_n)t}(x,y,z)$ tends towards $^t(a,b,c)$ for all $(x,y,z) \in [0,1]^3$. It is proved in [2] that any product of these three matrices, where all three occur, is a Pisot Matrix.
With each matrix $\mathbf{M}^{(i)}$ we associate a substitution σ_i as follows.

$$\sigma_1 : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{array} \qquad \sigma_2 : \begin{array}{l} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{array} \qquad \sigma_3 : \begin{array}{l} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{array}$$

and generalized substitutions Σ_{σ_i}. This S-adic system generates the so-called espisturmian words.

3.2 Generation of discrete planes

As proved in [6], the generalized substitution satisfies

$$(x_1, y_1, z_1) = f^{(i)}(x,y,z) \implies \Sigma_\sigma^{(i)}(\mathfrak{P}_{(x_1,y_1,z_1)}) \subset \mathfrak{P}_{(x,y,z)}$$

.

Since the unit cube $\mathcal{U}$ belongs to every discrete plane, we conclude

$$(x_n, y_n, z_n) = f^n(a,b,c) = f^{(i_n)} \dots f^{(i_1)}(a,b,c) \implies \Sigma_\sigma^{(i_1)} \dots \Sigma_\sigma^{(i_n)}(\mathcal{U}) \subset \mathfrak{P}_{(a,b,c)}$$

This is one example where the generalized continued fraction algorithm allows one to approximate some parts of a given discrete plane by a set of pieces obtained as the iteration of a given set of generalized substitutions.
This approximation is somewhat better (but on a smaller subset) than usual continued fractions, like Jacobi-Perron, since it is closer to be primitive. Note however that there exist also a degenerate case, that is, when the sequence of types contains no 1 (or, respectively, contains no 2 or no 3). Then we generate discrete lines in the discrete plane. Note also that the behavior seems to be bad when there are large powers of the same matrix in the expansion.
As noted above, one of the main questions is to know when this process covers all the discrete plane $\mathfrak{P}_{(a,b,c)}$. We hoped to be able to solve this problem easily at least in the case of bounded partial quotients, when all three types of substitutions occur in any sequence of bounded length in the expansion; however, the problem turned out to be much more delicate than we expected, with a combinatorial explosion. Let us notice some of the progresses and difficulties.
We consider the case of bounded partial quotient of order 4: any substitution occurs in any word of length 4. Up to renumbering the substitutions, there are six possible sequences, $1123, 1213, 1231, 1223, 1232, 1233$, giving rise to six substitutions σ_{1123} and so on.

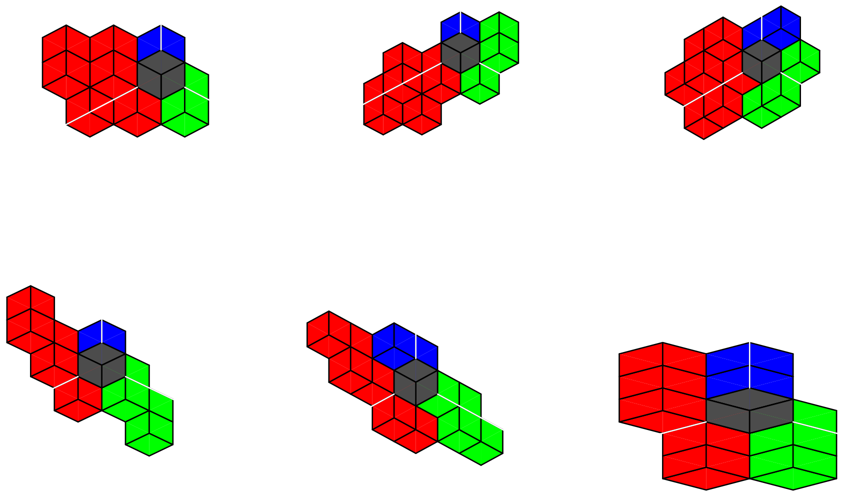

Figure 2: The images of the fundamental patch by the substitutions

As the following figure 2 shows, these substitutions all satisfy the generation property.

But, they do not satisfy the surrounding property; we show below the image of some neighborhood of the faces F_2 and F_3 by the substitution σ_{1231}; we see clearly that it is not a neighborhood of order 1 of that face.

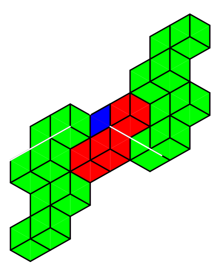

Figure 3: The image of E_2 by the substitution σ_{1231}

However, all examples show that, when we iterate the basic patch, then we get larger and larger parts of the plane, but at the moment it seems to entail the study of a very large number of cases.

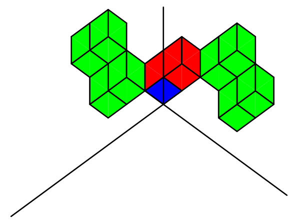

Figure 4: The image of E_3 by the substitution σ_{1231}

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