Extensions of S1S and the Composition Method

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Abstract

The study of the decidability of the monadic second-order (MSO) theory of $(\omega, <)$ has been carried out following two different approaches: the automata-theoretic approach (Büchi automata on infinite words) and the model-theoretic one (Shelah's composition method). A comparison of the two methods, pointing out the often overlooked merits of the second one, has recently been done by Thomas [20]. In this paper, we continue along this line showing that the composition method can be successfully exploited to decide the MSO theories of meaningful expansions of $(\omega, <)$. We first introduce the class of ultimately type-periodic unary predicates and we take advantage of the composition method to prove that the MSO theory of the expansion of $(\omega, <)$ with a predicate in this class is decidable. Next, we show that such a class coincides with Carton and Thomas' class of profinitely ultimately periodic predicates, which includes unary morphic predicates. Then, we define suitable conditions on morphic predicates, called linearity conditions, which allow one to apply the composition method to expansions of $(\omega, <)$ with morphic predicates of higher arity. As an application of this result, we provide an alternative proof of the decidability of the MSO theory of the expansion of $(\omega, <)$ with the binary predicate FLIP, which plays a major role in the study of hierarchical time structures involving different time granularities.

1 Introduction

The study of the decidability of the monadic second-order (MSO) theory of $(\omega, <)$ (such a theory is commonly known as the second-order theory of one successor, S1S for short) can be carried out following (at least) two different approaches: the automata-theoretic approach (Büchi automata on infinite words [1]) and the model-theoretic one (Shelah's composition method [5, 16]). A comparison of the two methods, pointing out the often overlooked merits of the second one, has been done by Thomas in [20]. In this paper, we continue along this line showing that the composition method can be successfully exploited to decide the MSO theories of meaningful expansions of $(\omega, <)$ with higher order predicates. As a preliminary step, we consider the case of unary predicates. We introduce the class of *ultimately type-periodic* predicates and we take advantage of the composition method to prove that the MSO theory of the expansion of $(\omega, <)$ with a predicate in this class is decidable. Next, we show that such a class coincides with Carton and Thomas' class of profinitely ultimately periodic predicates, which includes unary morphic predicates [2, 8]. Then, we define suitable conditions on morphic predicates, called linearity conditions, which allow one to apply the composition method to decide the MSO theory of expansions of $(\omega,<)$ with morphic predicates of higher arity. As an application of this result, we provide an alternative model-theoretic proof of the decidability of the MSO theory of the expansion of $(\omega,<)$ with the binary predicate FLIP, which plays a major role in the study of hierarchical time structures involving different time granularities [13].

S1S has been used for the formalization of properties of finite and infinite sequences. The decidability problem for S1S over infinite sequences plays a central role in computer science, as infinite sequences model the infinite computations of concurrent and reactive systems, a fundamental class of non-terminating programs [9, 10]. In [1], Büchi proves the decidability of S1S using concepts from automata theory. To this end, Büchi defines an (extremely natural) accepting condition for finite automata that allows one to characterize languages of infinite words. On this ground, he proves that the emptiness problem for such automata is decidable and he shows that the class of languages recognized by the now called *Büchi automata* is closed under Boolean operations and projection. Finally, taking advantage of these closure properties, he proves the decidability of S1S by showing that an S1S-formula can be effectively converted into a Büchi automaton, thereby providing an operational counterpart for S1S-formulas. The automata-theoretic approach has been used to prove the decidability of various extensions of S1S [18, 19]. An alternative, model-theoretic approach to the decidability problem for S1S is proposed by Shelah in [16]. This technique, called *composition method*, not only allows one to re-obtain Büchi's result, but it allows one to show much more, for example, on dense orders. Shelah's approach is more abstract than Büchi's one. The crucial ingredient of the composition method is the fact that it is possible to establish the *type* of a compound structure from the types of its components, in a computable manner, even in the case in which infinitely many components are involved. One of the merits of [20] is to have clearly identified the right amount of abstraction necessary to prove this fact (see Theorem 5 in [20]). We believe that Shelah's approach, as it already showed, has a great potential. In the following, we shall show that such a potential can be expressed in the treatment of logics whose expressiveness extends that of S1S by adding (higher arity) interpreted predicates.

In this paper, we explore the possibility of using the composition method to address the decision problem for MSO theories of expansions of $(\omega,<)$ with higher order morphic predicates. The decision problem for first-order theories of expansions of $(\omega,<)$ with unary and higher arity morphic predicates has been systematically investigated by Maes [8]. Unary morphic predicates are obtained as projections on the alphabet $\{0,1\}$ of morphic words, a morphic word being defined as the fixed point of a given morphism on words. Examples of unary morphic predicates are the Fibonacci predicate (consisting of all Fibonacci numbers) and the Thue-Morse word predicate (consisting of those numbers whose binary expansion has an even number of 1's) [2]. Maes shows that the first-order theory of the expansion of $(\omega,<)$ with a morphic predicate is decidable. To prove this result, he introduces a notion of morphism on multidimensional pictures, that generalizes the notion of morphism on words, and he shows that decidability of the first-order theory of the expansion of $(\omega,<)$ with a morphic predicate of any arity is guaranteed when the corresponding multidimensional morphic picture satisfies a special condition of *shape-symmetry*. The scheme of Maes' proof is the same as that of the proof of decidability by finite automata, but it deals with multidimensional morphisms instead of automata. In such a way, it gives a visual account of the central notion of shape-symmetry, which is only implicitly used in the automata setting. Maes' result basically states that shape-symmetry implies first-order decidability for morphic predicates of any arity. Since (the morphic word associated with) every unary morphic predicate is shape-symmetric, the decidability of the first-order theory of the expansion of $(\omega, <)$ with one such predicate immediately follows. This is not the case with morphic predicates of higher arity: there exist morphic predicates of arity 4 (which obviously do not satisfy the condition of shape-symmetry) whose first-order theory is undecidable [8]. It is not known whether or not morphic implies decidable for the first-order theories of morphic predicates of arity 2 or 3. The situation is much more involved in the second-order setting. As a matter of fact, Maes does not address the decision problem for the second-order theories of morphic predicates, which is only mentioned at the very end of the dissertation as a possible further development of his decidability results [8]. Unfortunately, there is no way to transfer his results, as they stand, from first-order theories to second-order ones. The Pascal triangle modulo 2 is an example of a morphic predicate of higher arity, that satisfies the condition of shape-symmetry (as shown by Maes in [8]), whose MSO theory is undecidable. In [2] Carton and Thomas prove that the transfer is possible in the restricted case of unary morphic predicates. By exploiting the automata-theoretic machinery, they show that it is possible to extend S1S with a unary morphic predicate preserving decidability. To the best of our knowledge, there exist no transfer results for morphic predicates of higher arity.

In the following, we show that Shelah's composition method can be exploited to lift decidability results for the theories of unary and higher arity morphic predicates from first-order to second-order. We first consider unary predicates and we show that the decidability of S1S extensions with ultimately type-periodic predicates (a class of unary predicates that includes morphic ones) can be easily established by applying the composition method. Then, we deal with the more difficult case of higher arity (morphic) predicates. At first sight the composition method seems to highly rely on the fact that the component structures share only $<$ as (interpreted) higher arity predicate. However, we shall show that, when suitable (and natural) *closure* conditions are satisfied, the sharing of a further higher arity predicate is possible. More precisely, we identify a set of conditions on morphic predicates of higher arity, called *linearity* conditions, that allow one to apply the composition method, and then we prove that any extension of S1S with a linear morphic predicate is decidable. We conclude the paper by applying such a result to the function *flip* introduced by Montanari et al. for the study of *time granularity* in [12]. For any natural number x, *flip*(x) is the number whose binary code is obtained by *flipping* to 0 the least significant 1 in the binary code of x. For instance, *flip*$(18) = 16$, since $18 = 1 \cdot 2^4 + 1 \cdot 2^1$ and $16 = 1 \cdot 2^4 + 0 \cdot 2^1$, while *flip*$(16) = 0$, since $16 = 1 \cdot 2^4$ and $0 = 0 \cdot 2^4$. The structure of *flip* is depicted in Figure 1. In [13], Montanari et al. prove that the extension of the first-order fragment of S1S with *flip* is the counterpart of a temporal logic for time granularity, called $UUTL$, which properly extends the well-known linear-time propositional temporal logic LTL. The full MSO theory of $(\omega, <, \mathit{flip})$, S1S(*flip*) for short, has been studied using automata theory by Monti and Peron in [14]. First, they extend the definition of systolic automata, introduced by Culik II, Salomaa, and Wood [4], to allow them to accept infinite words. Then, they prove that the class of systolic tree ω-languages (languages on infinite words) is a proper extension of the class of ω-languages recognized by Büchi automata and it maintains the same nice closure properties. Finally, they provide a logical counterpart to the systolic binary tree ω-languages by extending S1S with *flip* and they show that S1S(*flip*) is decidable. Another automata-theoretic proof of the decidability of S1S(*flip*), which takes advantage of unfolding and MSO interpretation, has been

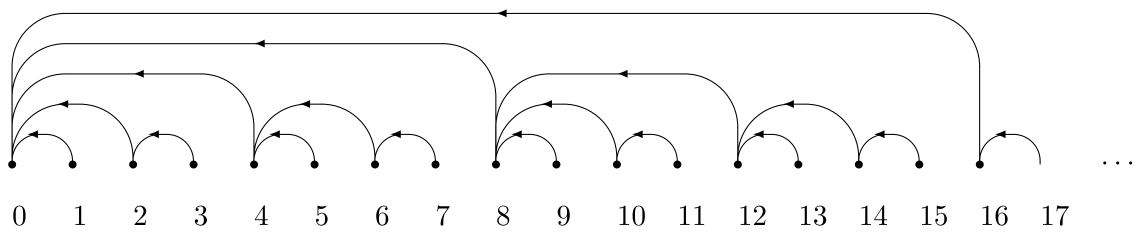

Figure 1: The structure of the function *flip*.

recently given in [21]. Here we provide an alternative model-theoretic proof of the decidability of S1S(*flip*) based on the composition method. First, we show that the graph of the function *flip* is a morphic relation (hereafter, with no changes in expressive power, we replace the function *flip* by its graph, the binary relation *FLIP*). Then, we show that the relation *FLIP* satisfies the linearity conditions, from which the decidability of S1S(*FLIP*) immediately follows.
The rest of the paper is organized as follows. In Section 2 we provide some background knowledge about Shelah's composition method (a summary of the automaton-based and the model-theoretic approaches to the decidability problem for $MSO[<]$, pointing out differences and similarities between the two proof methods, can be found in [11]). Section 3 deals with extensions of S1S with unary predicates. The class of *ultimately type-periodic* unary predicates is defined and the decidability of extensions of S1S with a predicate in this class is established. Then, we prove that this class coincides with Carton and Thomas' class of profinitely ultimately periodic words [2], which contains morphic unary predicates. In Section 4, we first introduce a restricted notion of binary morphic predicates and we show that the graph of the *FLIP* relation is morphic; then, we define suitable *linearity conditions* on morphic predicates, which allow one to apply the composition method to extension of S1S with predicates of higher arity. Finally, in Section 5 we prove that the *FLIP* predicate is linear. Conclusions provide an assessment of the work and outline future research directions.

2 Background

In this section we introduce the notation and provide some prerequisites, most of them connected to Shelah's composition method. We take as basic signature the set $\{<, =, P_1, \dots, P_m\}$, where $P_1, \dots, P_m$ are unary relation symbols. Formulas of the MSO language in this signature, denoted $MSO[<]$, are generated from atomic formulas of the forms $x = y$, $x < y$, $x \in P_i$, and $x \in Y$, where lower case variables are first-order variables and upper case variables are MSO variables, by Boolean connectives and the quantifiers $\exists$ and $\forall$ (applied to both kinds of variables). Following Thomas [20], we consider an equally expressive variant of $MSO[<]$, interpreted over ω, where only MSO variables are allowed and the set of atomic formulas is the following one:

- $NE(X \cap Y)$ (the intersection between X and Y is not empty);
- $X \subseteq Y$ (X is a subset of Y);

- $X < Y$ (there is an element in X which is smaller than an element in Y);
- $X_1 \cup \ldots \cup X_k = ALL$ (the union of the sets $X_1, \ldots, X_k$ is the universe of the structure).

We focus our attention on structures of the form $\mathcal{A} = (A, <)$, where $A \subseteq \omega$ and $<$ is the usual order over ω. $\mathcal{A}$ can be expanded by a tuple $\overline{P} = (P_1, \ldots, P_m)$ of subsets of A. Let $(\mathcal{A}, \overline{P})$ be the resulting structure. If $\phi(X_1, \ldots, X_m)$ is a formula with (at most) the indicated free-variables, we write $(\mathcal{A}, \overline{P}) \models \phi(X_1, \ldots, X_m)$ if ϕ is satisfied when interpreting X_i with P_i for all i.
A major role in the considered approach is played by the following notion of *complexity* of a formula. Every formula $\varphi(X_1, \ldots, X_m)$ can be written in prenex normal form as $Q_n\overline{Y_n} \ldots Q_1\overline{Y_1}\Psi(\overline{X}, \overline{Y_1} \ldots \overline{Y_n})$, where Q_i is either $\exists$ or $\forall$, $\overline{Y_i}$ is a tuple of k_i set variables, Ψ is quantifier free, and $\overline{X}$ are the only free variables in φ. The complexity of φ is the tuple $\overline{k} = (k_1, \ldots, k_n)$ of the lengths of the quantifier blocks. It does not simply record the nesting of quantifiers, but it also keeps track of the quantifier alternation depth. A formula φ of complexity $\overline{k}$ is said to be a $\overline{k}$-formula. Given two structures $(\mathcal{A}, \overline{P})$ and $(\mathcal{B}, \overline{Q})$, we write $(\mathcal{A}, \overline{P}) \equiv_{\overline{k}} (\mathcal{B}, \overline{Q})$ if the two structures satisfy the same $\overline{k}$-formulas.
A $\overline{k}$-type is a set of $\overline{k}$-formulas that allows one to identify structures satisfying the same $\overline{k}$-formulas. A $\overline{k}$-type is built up starting from two parameters: the complexity $\overline{k}$ and the number m of admitted free variables.

Definition 1. *The set $\mathcal{T}^{\overline{k}}(m)$ of all possible $\overline{k}$-types is inductively defined as follows:*

$$\begin{cases} \mathcal{T}^{\lambda}(m) := \mathcal{P}(Atom_m); \\ \mathcal{T}^{\overline{k}.k_{n+1}}(m) := \mathcal{P}(\mathcal{T}^{\overline{k}}(m + k_{n+1})), \end{cases}$$

where $Atom_m$ is the set of all atomic formulas with m variables.
The $\overline{k}$-type of a structure $\mathcal{A}$, expanded with m unary predicates $P_1, \ldots, P_m$, is a set belonging to $\mathcal{T}^{\overline{k}}(m)$ inductively defined as follows:

$$\begin{cases} T^{\lambda}(\mathcal{A}, \overline{P}) := \{\varphi(\overline{X}) | \varphi(\overline{X}) atomic, (\mathcal{A}, \overline{P}) \models \varphi(\overline{X})\}; \\ T^{\overline{k}.k_{n+1}}(\mathcal{A}, \overline{P}) := \{T^{\overline{k}}(\mathcal{A}, \overline{P}, \overline{Q}) | \overline{Q} = (Q_1, \ldots, Q_{k_{n+1}}) \in \mathcal{P}(A)^{k_{n+1}}\}, \end{cases}$$

where $\overline{X}$ stands for a fixed sequence of variables $X_1, \ldots, X_m$.

One can show that two structures with the same $\overline{k}$-type satisfy the same $\overline{k}$-formulas.
The relationship between the decidability of the MSO theory of an expanded structure $(\mathcal{A}, \overline{P})$ and its $\overline{k}$-types is expressed by the following proposition.

Proposition 2. *The MSO theory of a labelled order $(\mathcal{A}, \overline{P})$ is decidable if and only if there exists a computable function that, for each $\overline{k}$, returns the set $T^{\overline{k}}(\mathcal{A}, \overline{P})$.*

Given a sequence of disjoint structures $(\mathcal{A}_i, \overline{P}_i)_{i \in I}$, where I is either a finite ordered set or ω, it is possible to show that the $\overline{k}$-type of the ordered sum $\sum_{i \in I}(\mathcal{A}_i, \overline{P}_i) := (\bigcup_{i \in I} \mathcal{A}_i, <, \bigcup_{i \in I} \overline{P})$, where $a < b$ iff $a <^{\mathcal{A}_i} b$, for some $i \in I$, or $a \in \mathcal{A}_i, b \in \mathcal{A}_j$, and $i < j$, and $\cup_{i \in I}\overline{P} := (\cup_{i \in I} P_{i,1}, \ldots, \cup_{i \in I} P_{i,m})$, only depends on the $\overline{k}$-types of the components. The sum of types is defined accordingly. If τ and σ are the $\overline{k}$-types of the structures $(\mathcal{A}_1, \overline{P}_1)$ and $(\mathcal{A}_2, \overline{P}_2)$, respectively, then their sum $\tau \oplus \sigma$ is the $\overline{k}$-type of the structure $(\mathcal{A}_1, \overline{P}_1) + (\mathcal{A}_2, \overline{P}_2)$.

We conclude the section with Shelah's composition theorem. Given a finite alphabet A, we denote by A^* (resp., A^+) the set of finite words on A (resp., non-empty finite words). A finite (resp., infinite) A-word w, viewed as a function from the length $l(w)$ of w (resp., ω) to A, can be interpreted as a structure for the language $\{<\} \cup \{P_a : a \in A\}$ as follows: the domain of the structure and the order relation are given by the set $\{0, \ldots, l(w)-1\}$ (resp., ω) with its natural order, and $i \in P_a$ if and only if $w(i) = a$, for all $a \in A$. Thus, given an A-word w, we can define its $\overline{k}$-type on $|A|$ parameters.
For any complexity $\overline{k}$ and natural number m, let $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ be an enumeration of the finite $\overline{k}$-types, that is, of the types of finite structures, on m parameters. Shelah's composition theorem can be stated as follows (see Theorem 5 in [20]).

Theorem 3. *(Shelah's Composition Theorem) Given a complexity $\overline{k}$ and a natural number m, one can compute a complexity $\overline{r}$ such that the $\overline{k}$-type of an ordered sum $\sum_{i \in I}(\mathcal{A}_i, \overline{P}_i)$ of finite linear orders extended with m parameters, where I is either a finite ordered set or ω, can be determined (and computed from) the $\overline{r}$-type of the Σ-word $\rho_0\rho_1\ldots$, where $\rho_i = T^{\overline{k}}((\mathcal{A}_i, \overline{P}_i))$, for all $i \in I$.*

3 $MSO[<]$ extensions with ultimately type-periodic predicates

In this section we introduce the class of *ultimately type-periodic* unary predicates and we apply Shelah's composition method to prove the decidability of the MSO theory of the expansion of $(\omega, <)$ with a predicate in this class. Then, we show that this class includes that of profinitely ultimately periodic (unary) predicates. The class of profinitely ultimately periodic predicates, which in its turn includes the class of morphic unary predicates, was identified by Carton and Thomas in [2], where the decidability of the MSO theory of the expansion of $(\omega, <)$ with a predicate in such a class was proved by an automaton-based argument.
The class of ultimately type-periodic words (predicates) is defined as follows.

Definition 4. *An infinite sequence of finite words $(w_i)_{i\in\omega}$ on a finite alphabet A is said to be* effectively ultimately type-periodic *if there exists a computable function that, given a complexity $\overline{k}$, returns two integers n, p, with $p > 0$, such that, for any $h \geq n$, it holds that $w_h \equiv_{\overline{k}} w_{h+p}$, where any A-word can be viewed as a structure for the language $\{<\} \cup \{P_a : a \in A\}$. An infinite word w is said to be* ultimately type-periodic *if it can be factorized as $w = w_0w_1\ldots$, where the sequence $(w_i)_{i\in\omega}$ is ultimately type-periodic.*

Theorem 5. *An ultimately type-periodic word w has a decidable MSO theory.*

Proof. Let $w = w_0w_1\ldots$, where the sequence $(w_i)_{i\in\omega}$ is ultimately type-periodic. Taking advantage of Proposition 2, we prove the theorem by showing that there eixts a computable function that, given $\overline{k}$, returns the $\overline{k}$-type of w. For any given $\overline{k}$, let $n, p > 0$ be the two integers of Definition 4, and let:

$$w^* := w_0w_1\ldots w_{n-1}, \qquad w'_0 := w_nw_1\ldots w_{n+p-1}, \qquad w'_1 := w_{n+p}\ldots w_{n+2p-1},$$
$$\ldots, \qquad w'_i := w_{n+ip}\ldots w_{n+(i+1)p-1}, \qquad \ldots$$

Then, $w = w^* w'_0 \dots w'_i \dots$, and, by Theorem 3, w'_i, w'_j have the same $\overline{k}$-type for all i, j. Hence, we have $w = w^* w'$, where $w' = w'_0 \dots w'_i \dots$.
By Theorem 3, we have that the $\overline{k}$-type ν of w' is computable from the $\overline{r}$-type μ of a constant Σ-word, and μ is easily computable from the $\overline{r}$-type of $(\omega, <)$ alone (see Corollary 7 of [20] for a proof of this fact). In this way, we effectively reduced the computability of the $\overline{k}$-type of w' to the computability of the $\overline{r}$-type of $(\omega, <)$. The complexity $\overline{r}$ is obtained effectively from $\overline{k}$, and the $\overline{r}$-type of $(\omega, <)$ is computable, e.g., by using the algorithm described in [20]. Hence, we have an effective procedure to compute the $\overline{k}$-type of w'.
Once we have the $\overline{k}$-type of this structure, we can use Theorem 3 once more, this time relatively to the decomposition $w = w^* w'$, to compute the $\overline{k}$-type of w from the $\overline{r}$-type of the word $\rho_0 \rho_1$, with $\rho_0 = T^{\overline{k}}(w^*)$ and $\rho_1 = T^{\overline{k}}(w')$. □

We now show that the class of profinitely ultimately periodic words is included in the class of ultimately type-periodic words. According to [2], given a finite alphabet A, a sequence $(w_i)_{i \in \omega}$ of finite A-words is said to be *effectively profinitely ultimately periodic* if for any morphism ϕ from A^+ into a finite semigroup S, one can effectively compute two integers n, p, with $p > 0$, such that, for any $h \geq n$, $\phi(w_h) = \phi(w_{h+p})$. An infinite word w is said to be *profinitely ultimately periodic* if it can be factorized as $w = w_0 w_1 \dots$, where the sequence $(w_i)_{i \in \omega}$ is effectively profinitely ultimately periodic.

Theorem 6. *Any effectively profinitely ultimately periodic word is effectively ultimately type-periodic.*

Proof. Let $(w_i)_{i \in \omega}$ be an effectively profinitely ultimately periodic sequence and let $m = |A|$. Moreover, let $T^{\overline{k}}(Fin(m))$ be the set of $\overline{k}$-types of finite structures on m parameters and let ϕ be the function from A^+ to $T^{\overline{k}}(Fin(m))$ defined by $\phi(w) := \mathcal{T}^{\overline{k}}(w)$. By Theorem 3, ϕ is a semigroup morphism from the semigroup of finite words with concatenation and the finite semigroup $(\mathcal{T}^{\overline{k}}(Fin(m)), \oplus)$, where $\oplus$ denotes the ordered sum of types. Since $(w_i)_{i \in \omega}$ is an effectively profinitely ultimately periodic sequence, one can effectively compute two integers $n, p > 0$ such that, for any $h \geq n$, $\phi(w_h) = \phi(w_{h+p})$, that is, $w_h \equiv_{\overline{k}} w_{h+p}$, and thus the word $(w_n)_{n \in \omega}$ is effectively ultimately type-periodic. □

Corollary 7. *If w is a profinitely ultimately periodic word, then w has a decidable MSO theory.*

Proof. It immediately follows from Theorems 5 and 6. □

Recently, Rabinovich has shown that any unary predicate P such that the MSO theory of the expansion of $(\omega, <)$ with P is decidable belongs to the class of profinitely ultimately periodic predicates [15]. This implies that the class of effectively ultimately type-periodic predicates in fact coincides with that of profinitely ultimately periodic ones.

4 Morphic predicates of higher arity and the composition method

In this section we focus our attention on morphic predicates of higher arity. The generalization of the notion of morphic predicate from unary to higher arity predicates was given by Maes in

[8]. For the sake of simplicity, we restrict ourselves to binary predicates and we simplify Maes' definitions as follows.
Given a finite alphabet A, a d-dimensional A-square D is a function from d^2 $(= d \times d)$ to A, where d is either ω or $\{0, \dots, d-1\}$. We denote the set of d-dimensional squares by $Sq_d(A)$. Any $D \in Sq_d(A)$ is identified with a structure for the language $\{<\} \cup \{R_a : a \in A\}$: the domain of D and the relation $<$ are given by either ω or the set $\{0, \dots, d-1\}$ with their natural order, while $R_a^D = \{(h,k) \in d^2 : D(h,k) = a\}$. The expansion of the structure D with m subsets $P_1, \dots, P_m$ of its domain is denoted by $(D, P_1, \dots, P_m)$, or simply by $(D, \overline{P})$.
We define a *morphism* as a function ϕ from A to A-squares of the same dimension d. Such a notion is more restrictive than that of multidimensional morphism given by Maes, because it constrains images to be squares of the same dimension, while Maes' one encompasses also *rectangular* images. Thanks to this simplification, however, we do not need to impose any additional condition on the definition of morphism to guarantee that squares can be composed. The image of ϕ on a finite square D is the finite square obtained by substituting each letter a in D by $\phi(a)$.

Example 8. *Let $A = \{a, b, c\}$ and $\phi_0 : A \to A^{*^2}$ be the function defined as follows:*

$$\phi_0 : \left\{ \quad a \mapsto \begin{array}{|c|c|} \hline a & b \\ \hline c & a \\ \hline \end{array} \qquad b \mapsto \begin{array}{|c|c|} \hline b & c \\ \hline c & c \\ \hline \end{array} \qquad c \mapsto \begin{array}{|c|c|} \hline c & c \\ \hline c & c \\ \hline \end{array} \right.$$

We have that:

$$\phi_0(a) = \begin{array}{|c|c|} \hline a & b \\ \hline c & a \\ \hline \end{array} \qquad \phi_0(\phi_0(a)) = \phi_0^2(a) = \begin{array}{|c|c|c|c|} \hline a & b & b & c \\ \hline c & a & c & c \\ \hline c & c & a & b \\ \hline c & c & c & a \\ \hline \end{array}$$

$$\phi_0^3(a) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline a & b & b & c & b & c & c & c \\ \hline c & a & c & c & c & c & c & c \\ \hline c & c & a & b & c & c & c & c \\ \hline c & c & c & a & c & c & c & c \\ \hline c & c & c & c & a & b & b & c \\ \hline c & c & c & c & c & a & c & c \\ \hline c & c & c & c & c & c & a & b \\ \hline c & c & c & c & c & c & c & a \\ \hline \end{array}$$

One can easily see that, for any natural number n, the square $\phi_0^{n+1}(a)$ is obtained from the square $\phi_0^n(a)$ by duplicating it along the diagonal and filling in all remaining positions with the value c, except for the position $(0, 2^n)$ whose value is b. Since $\phi_0^n(a)$ is an initial factor of $\phi_0^{n+1}(a)$, it makes sense to consider the limit $\phi_0^\omega(a) = lim_{n \to \infty} \phi_0^n(a)$, and the infinite square $\phi_0^\omega(a)$ is a fixed point of the morphism ϕ_0.

As a general rule, if ϕ is a morphism and the letter associated with position $(0,0)$ of $\phi(a)$ is a, then $\phi^\omega(a) = lim_{n \to \infty} \phi^n(a)$ exists and it is a fixed point of ϕ. The notion of morphic predicate can thus be generalized from unary to binary predicates as follows.

Definition 9. *A binary predicate R on ω is said to be* morphic *if there exists a morphism $\phi : A \to Sq_d(A)$, a letter $a \in A$, and a projection $\psi : A \to \{0,1\}$ such that the structure $(\omega, <, R)$ is isomorphic to the square $\psi(\phi^\omega(a))$.*

As an example, we show that the predicate FLIP is morphic. Consider the morphism ϕ_0 of Example 8 and the projection function $\psi : A \to \{0,1\}$ defined as follows:

$$\psi := \begin{cases} a \mapsto 0 \\ b \mapsto 1 \\ c \mapsto 0 \end{cases}$$

We prove that the structure $\psi(\phi_0^\omega(a))$ is isomorphic to $(\omega, <, \mathrm{FLIP})$.

Proposition 10. *The infinite square $\psi(\phi_0^\omega(a))$ is isomorphic to the structure* $(\omega, <, \mathrm{FLIP})$.

Proof. As a preliminary remark, we notice that the interval $[0, 2^n)$ is closed under the FLIP relation and that $(\omega, <, \mathrm{FLIP})$ is isomorphic to the limit of the sequence of substructures $([0, 2^n), <, \mathrm{FLIP}|_{[0,2^n)})_{n\in\omega}$. Since $\psi(\phi_0^\omega(a))$ is the limit of the sequence $(\psi(\phi_0^n(a)))_{n\in\omega}$ and the dimension of the square $\psi(\phi_0^n(a))$ is 2^n, to prove the thesis it suffices to show that the FLIP relation restricted to the ordered interval $[0, 2^n)$ is isomorphic to the finite square $\psi(\phi_0^n(a))$. This can be proved by induction on n.

- The basic case $n = 0$ is trivial.
- As for the inductive step, by definition we have that the FLIP relation over the interval $[0, 2^{n+1})$ can be obtained from the FLIP relation over the interval $[0, 2^n)$: first, by taking advantage of the ordered bijection between $[0, 2^n)$ and $[2^n, 2^{n+1})$, we copy the FLIP relation from $[0, 2^n)$ to $[2^n, 2^{n+1})$; then we add the pair $(2^n, 0)$ to the relation FLIP. In a similar way, the square $\psi(\phi_0^{n+1}(a))$ can be obtained from the square $\psi(\phi_0^n(a))$ by duplicating it along the diagonal and by putting 0 on all other positions, except for the position $(0, 2^n)$ where the value is 1. By using the inductive hypothesis, we can show that the FLIP relation over the interval $[0, 2^{n+1})$ is isomorphic to the square $\psi(\phi_0^{n+1}(a))$.

□

Since the proposed restricted notion of morphism satisfies the condition of shape-symmetry, from Maes' result [8] it follows that if R is a morphic binary predicate, then the first-order theory of $(\omega, <, R)$ is decidable. However, the problem of deciding the MSO theory of a morphic binary predicate is much more complicated and there are simple cases of shape-symmetric morphic predicates whose MSO theory is undecidable. As already mentioned, an interesting example of undecidability is that of the Pascal triangle modulo 2, which can be defined as the fixed point $\varphi^\omega(1)$ of the following morphism:

$$\varphi : \left\{ 1 \mapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline\end{array} \quad 0 \mapsto \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline\end{array} \right.$$

One can easily see that $\varphi^{n+1}(1)$ is obtained from $\varphi^n(1)$ by duplicating it on the right and underneath, and then filling all remaining positions with 0's. It is possible to prove (see Korec's work [7]) that the first-order theory of $(\omega, +, \times)$ can be interpreted in the weak MSO theory of the Pascal triangle modulo 2, which implies that the MSO theory of the Pascal triangle modulo 2 is undecidable.

We now provide suitable conditions on morphic predicates of higher arity that allow one to apply Shelah's composition method to prove the decidability of the MSO theory of their square. In the next section, we shall prove that the FLIP predicate satisfies these conditions.
For any given complexity $\overline{k}$, let $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ be a fixed enumeration of the finite $\overline{k}$-types.

Definition 11. *(Linearity conditions) Let ϕ be a multi-dimensional morphism over an alphabet A and let a be an element of A such that $\phi^\omega(a)$ is a fixed point. Moreover, let $a, a_1, \dots, a_s$ be the letters we read on the diagonal of the square $\phi(a)$. The fixed point $\phi^\omega(a)$ is said to be* linear *if for any complexity $\overline{k}$, we can effectively compute a complexity $\overline{r}$ such that:*

1. *the $\overline{k}$-type of $\phi^\omega(a)$ only depends on, and can be calculated from, the $\overline{r}$-type of the infinite Σ-word $\rho_0\rho_{1,1}\dots\rho_{1,s}\dots\rho_{n,1}\dots\rho_{n,s}\dots$ defined by:*

   $$\rho_0 = T^{\overline{k}}(\phi(a)) \text{ and } \rho_{n,i} = T^{\overline{k}}(\phi^n(a_i)) \text{ for } n \geq 1, i \in \{1, \dots s\};$$

2. *for any natural number n and $i \in \{1, \dots, s\}$, the $\overline{k}$-type of $\phi^{n+1}(a_i)$ only depends on the $\overline{k}$-type of $\phi^n(a_i)$.*

Theorem 12. *The MSO theory of any linear fixed point $\phi^\omega(a)$ is decidable.*

Proof. To prove the thesis it suffices to show that the infinite Σ-word $\rho = \rho_0\rho_{1,1}\dots\rho_{1,s}\dots\rho_{n,1}\dots\rho_{n,s}\dots$ of Definition 11 has a decidable MSO theory. This can be done as follows.
First, we find two natural numbers $n, m \geq 1$ such that, for all $i \in \{1, \dots, s\}$, it holds that $\phi^n(a_i) \equiv_{\overline{k}} \phi^{n+m}(a_i)$. These two number can be determined in the following way. Since the set of $\overline{k}$-types is finite, for every $i \in \{1, \dots, s\}$, there are $n_i, m_i \geq 1$ such that $\phi^{n_i}(a_i) \equiv_{\overline{k}} \phi^{n_i+m_i}(a_i)$. Then, by the second linearity property, we have that, for all natural numbers j, it holds that $\phi^{n_i+j}(a_i) \equiv_{\overline{k}} \phi^{n_i+m_i+j}(a_i)$. Hence, if n is the maximum of all the n_i, we have that $\phi^n(a_i) \equiv_{\overline{k}} \phi^{n+m_i}(a_i)$. Another application of the second linearity property gives $\phi^n(a_i) \equiv_{\overline{k}} \phi^{n+hm_i}(a_i)$, for any natural number h. Hence, if m is a common multiple of all the m_i's, we have that $\phi^n(a_i) \equiv_{\overline{k}} \phi^{n+m}(a_i)$ for all $i \in \{1, \dots, s\}$.
By the second linearity property, this implies that, for all j, $\phi^{n+j}(a_i) \equiv_{\overline{k}} \phi^{n+mj}(a_i)$. As a consequence, the Σ-word ρ can be decomposed into a finite Σ-word $\rho' = \rho_0\rho_{1,1}\dots\rho_{1,s}\dots\rho_{n-1,1}\dots\rho_{n-1,s}$ and an infinite Σ-word ρ'' which is periodic with period equal to the finite word $\rho_{n,1}\dots\rho_{n,s}$ $\rho_{n+1,1}\dots\rho_{n+1,s}\dots\rho_{n+m-1,1}\dots\rho_{n+m-1,s}$.
Hence, ρ has a decidable MSO theory and the first linearity property implies the decidability of the MSO theory of the fixed point $\phi^\omega(a)$. $\square$

5 Linearity of the FLIP predicate

In this section we prove that the fixed point $\phi_0^\omega(a)$ that defines the FLIP predicate (see Section 4) is linear. Given the inductive structure of the proof, we actually need to prove a stronger result involving *parameters*.

In the proof of the theorem as well as in that of the following corollaries, it will be useful to keep in mind that the square $\phi_0^{n+1}(a)$ is obtained from the square $\phi_0^n(a)$ by copying the square $\phi_0^n(a)$ along the diagonal and filling in the remaining positions with c's, except for one b to be put in position $(0, 2^n)$.

Once more, instead of working with the standard MSO language over the signature $\{<, R_a, R_b, R_c\}$, we add a constant $first$ to the language, which will be interpreted as the first element of the structures, and consider an expressively equivalent logic (over $(\omega, <)$) where only MSO variables $X_1, X_2, \ldots$ are allowed and the set of atomic formula is the following one:

$$\begin{array}{cc} \{first\} \subseteq X_i; & X_i \subseteq \{first\}; \\ X_i \subseteq X_j; & X_i \subseteq X_{i_1} \cup \ldots \cup X_{i_n}; \\ X_{i_1} \cup \ldots \cup X_{i_n} = ALL; & NE(X_i \cap X_j); \\ X_i < X_j; & R_e(X_i, X_j), \text{ for } e = a, b, c. \end{array}$$

Formulas with free variables in $\{X_1, \ldots, X_m\}$ are interpreted in a natural way in a structure $(\mathcal{M}, \overline{P})$, where $\mathcal{M}$ is an interpretation of the language $\{<, first, R_a, R_b, R_c\}$ and $\overline{P} = (P_1, \ldots, P_m)$ are interpretations of the MSO variables. In particular (the other cases are known or similar), we have:

$$\begin{array}{lcl} (\mathcal{M}, \overline{P}) \models \{first\} \subseteq X_i & \Leftrightarrow & \text{the interpretation of} \\ & & first \text{ belongs to } P_i; \\[1em] (\mathcal{M}, \overline{P}) \models NE(X_i \cap X_j) & \Leftrightarrow & P_i \cap P_j \neq \emptyset; \\[1em] (\mathcal{M}, \overline{P}) \models X_i < X_j & \Leftrightarrow & \exists u \in P_i, \exists v \in P_j \text{ such that} \\ & & u < v \text{ is true in } \mathcal{M}; \\[1em] (\mathcal{M}, \overline{P}) \models R_a(X_i, X_j) & \Leftrightarrow & \exists u \in P_i, \exists v \in P_j \text{ such that} \\ & & R_a(u, v) \text{ is true in } \mathcal{M}. \end{array}$$

In the following, the word "type" always refers to a type in this language. As before, if $\overline{k}$ is a complexity and m is a natural number, we denote by $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ a fixed enumeration of the $\overline{k}$-types on m parameters.

Theorem 13. *Given a type $\overline{k}$, we can effectively determine a complexity $\overline{r}$ in such a way that the $\overline{k}$-type of the infinite square $(\phi_0^\omega(a), \overline{P})$ can be computed from the $\overline{r}$-type of the infinite word $\rho_0 \ldots \rho_n \ldots$ over the alphabet Σ, where*

$$\rho_0 = T^{\overline{k}}((\phi_0(a), \overline{P} \cap [0, 2))); \rho_i = T^{\overline{k}}(\phi_0^i(a), \overline{P} \cap [2^i, 2^{i+1})), \text{ for } i > 0.$$

Proof. The proof follows the same lines of the proof of the Composition Theorem given in [20]. First, we give an inductive definition of the complexity $\overline{r} = f(\overline{k}, m)$, by means of a computable function f:

$$\begin{cases} f(\lambda, m) := \lambda \\ f(\overline{k}.k_{n+1}, m) := f(\overline{k}, m + k_{n+1}).|\mathcal{T}^{\overline{k}}(m + k_{n+1})|, \end{cases}$$

where $\mathcal{T}^{\overline{k}}(m + k_{n+1})$ is the finite set of all $\overline{k}$-types over $m + k_{n+1}$ parameters.
We now proceed by induction on the length n of the complexity $\overline{k}$.
Case $n = 0$. Here $\overline{k} = \lambda$. What we have to check is that the membership of an atomic formula to the λ-type of the square $(\phi_0^\omega(a), \overline{P})$ only depends on, and can be computed from, the λ-type of the infinite word $\rho_0 \dots \rho_i \dots$ over the alphabet Σ, where $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is the set of λ-types over m parameters and

$$\rho_0 = T^\lambda((\phi_0(a), \overline{P} \cap [0, 2)); \rho_i = T^\lambda(\phi_0^i(a), \overline{P} \cap [2^i, 2^{i+1})), \text{ for } \ i > 0.$$

We check this for the atomic formulas of our language. Let μ be $\overline{r}$-type of the word $\rho_0 \dots \rho_i \dots$, viewed as a structure for the language $\{<, Q_1, \dots, Q_s\}$. We denote by $Y_1, \dots, Y_s$ the variables representing the predicates $Q_1, \dots, Q_s$ in μ. Let ν be the λ-type of $(\phi_0^\omega(a), \overline{P})$ and let us denote by $X_1, \dots, X_m$ the variables representing the predicates $P_1, \dots, P_m$ in ν. We have that:

- $(\{first\} \subseteq X_i) \in \nu \Leftrightarrow \exists z\ [\{first\} \subseteq Y_z) \in \mu\ \wedge\ (\{first\} \subseteq X_i) \in \sigma_z]$
  (this condition holds when $\{first\} \subseteq X_i$ is true in $(\phi_0(a), \overline{P} \cap [0, 2))$, that is, when $\{first\} \subseteq X_i$ belongs to ρ_0);
- $(X_i \subseteq \{first\}) \in \nu \Leftrightarrow \exists z\ [\{first\} \subseteq Y_z) \in \mu\ \wedge\ (X_i \subseteq \{first\}) \in \sigma_z]$;
- $(X_i \subseteq X_{i_1} \cup \dots \cup X_{i_n}) \in \nu \Leftrightarrow \forall z\ [NE(Y_z) \in \mu \rightarrow (X_i \subseteq X_{i_1} \cup \dots \cup X_{i_n}) \in \sigma_z]$;
- $(X_{i_1} \cup \dots \cup X_{i_n} = ALL) \in \nu \Leftrightarrow \forall z\ [NE(Y_z) \in \mu \rightarrow (X_{i_1} \cup \dots \cup X_{i_n} = ALL) \in \sigma_z]$;
- $NE(X_i \cap X_j) \in \nu \Leftrightarrow \exists z\ [NE(Y_z) \in \mu\ \wedge\ NE(X_i \cap X_j) \in \sigma_z]$;
- $(X_i < X_j) \in \nu$ if and only if one of the following conditions holds:
  1. $\exists z\ [NE(Y_z) \in \mu\ \wedge\ (X_i < X_j) \in \sigma_z]$;
  2. $\exists z \exists z' [(Y_z < Y_{z'}) \in \mu\ \wedge\ NE(X_i) \in \sigma_z\ \wedge\ NE(X_j) \in \sigma_{z'}]$.
- if $e \in \{a, b, c\}$, then $R_e(X_i, X_j) \in \nu$ if and only if one of the following conditions holds:
  1. $\exists z\ [NE(Y_z) \in \mu\ \wedge\ R_e(X_i, X_j) \in \sigma_z]$;
  2. if $e = b$, there are z, z' such that: $(\{first\} \subseteq Y_z) \in \mu$, $(\{first\} \subseteq X_i) \in \sigma_z$, $NE(Y_{z'}) \in \mu$, $(Y_{z'} \subseteq \{first\}) \notin \mu$, and $(\{first\} \subseteq X_j) \in \sigma_{z'}$ (this condition holds when X_i contains the first element in the structure $(\phi_0(a), (\overline{P}) \cap [0, 2))$ and X_j contains the first element in a structure $(\phi_0^i(a), \overline{P} \cap [2^i, 2^{i+1}))$, for $i > 0$);
  3. if $e = c$, then either there are z, z' such that $(Y_z < Y_{z'}) \in \mu$, $NE(X_i) \in \sigma_z$, $NE(X_j) \in \sigma_{z'}$, and $(X_j \subseteq \{first\}) \notin \sigma_{z'}$, or there are z, z' such that $(Y_z < Y_{z'}) \in \mu$, $NE(X_j) \in \sigma_z$, and $NE(X_i) \in \sigma_{z'}$.

Case $n > 0$. We want to compute the set $\mathcal{T}^{\overline{k}.k_{n+1}}(\phi_0^\omega(a), \overline{P})$ from the $f(\overline{k}.k_{n+1}, m)$-type of the infinite word $\rho_0 \rho_1 \dots$, where

$$\rho_0 = T^{\overline{k}.k_{n+1}}((\phi_0(a), \overline{P} \cap [0, 2)); \rho_i = T^{\overline{k}.k_{n+1}}(\phi_0^i(a), \overline{P} \cap [2^i, 2^{i+1})), \text{ for } \ 0 < i.$$

We have $f(\overline{k}.k_{n+1}, m) = \overline{r}.r_{n+1}$, where

$$\overline{r} = f(\overline{k}, m + k_{n+1}), \quad r_{n+1} = |\mathcal{T}^{\overline{k}}(m + k_{n+1})|,$$

and

$$\mathcal{T}^{\overline{k}.k_{n+1}}(\phi_0^\omega(a), \overline{P}) := \{\mathcal{T}^{\overline{k}}(\phi_0^\omega(a), \overline{P}, \overline{R}) : \overline{R} = (R_1, \ldots, R_{k_{n+1}}) \in \mathcal{P}(\omega)^{k_{n+1}}\}.$$

By induction, we know that for a fixed choice of a k_{n+1}-tuple of sets $\overline{R} = (R_1, \ldots, R_{k_{n+1}})$, we can compute $\mathcal{T}^{\overline{k}}(\phi_0^\omega(a), \overline{P}, \overline{R})$ by using the $\overline{r}$-type of the word $\xi_0^{\overline{R}} \xi_1^{\overline{R}} \ldots$, where

$$\xi_0^{\overline{R}} = T^{\overline{k}}((\phi_0(a), \overline{P} \cap [0, 2), \overline{R} \cap [0, 2)),$$

and, for $i > 0$,

$$\xi_i^{\overline{R}} = T^{\overline{k}}(\phi_0^i(a), \overline{P} \cap [2^i, 2^{i+1}), \overline{R} \cap [2^i, 2^{i+1}))).$$

Hence, to prove the thesis it suffices to show that the finite collection C of the $\overline{r}$-types of these words, obtained by varying $\overline{R}$, can be computed from the $\overline{r}.r_{n+1}$ type of the word $\rho_0\rho_1\ldots$. For a fixed $\overline{R}$, the $\overline{r}$-type of the word $\xi_0^{\overline{R}} \xi_1^{\overline{R}} \ldots$ is the set $\mathcal{T}^{\overline{r}}(\omega, <, Q_1^{\overline{R}}, \ldots, Q_{s'}^{\overline{R}})$, where $\{\tau_1, \ldots, \tau_{s'}\}$ is a fixed enumeration of the $\overline{k}$-types over $m + k_{n+1}$ parameters and $Q_h^{\overline{R}} = \{i \in \omega : \xi_i^{\overline{R}} = \tau_h\}$. On the other hand, the $\overline{r}.r_{n+1}$-type of the word $\rho_0\rho_1 \ldots$ is the set $\mathcal{T}^{\overline{r}.r_{n+1}}(\omega, <, Q_1, \ldots, Q_s)$, where $\{\sigma_1, \ldots, \sigma_s\}$ is a fixed enumeration of the $\overline{k}.k_{n+1}$-types over m parameters and $Q_h = \{i \in \omega : \rho_i = \sigma_h\}$. We have

$$\mathcal{T}^{\overline{r}.r_{n+1}}(\omega, <, Q_1, \ldots, Q_s) := \{\mathcal{T}^{\overline{r}}(\omega, <, Q_1, \ldots, Q_s, W_1, \ldots, W_{r_{n+1}}) : W_i \subseteq \omega\}.$$

Since $r_{n+1} = |\mathcal{T}^{\overline{k}}(m + k_{n+1})|$, the two numbers s' and r_{n+1} are equal. We have to find a criterion to select, among the elements of the set of types $\mathcal{T}^{\overline{r}.r_{n+1}}(\omega, <, Q_1, \ldots, Q_s)$, exactly those $\overline{r}$-types of expansions $(\omega, <, Q_1, \ldots, Q_s, W_1, \ldots, W_{r_{n+1}})$, where the predicates W_h are not arbitrary anymore, but they correspond to a tuple $\overline{R}$ such that $W_h = Q_h^{\overline{R}} = \{i \in \omega : \xi_i^{\overline{R}} = \tau_h\}$. Once we have the $\overline{r}$-type of the expansion $(\omega, <, Q_1, \ldots, Q_s, W_1, \ldots, W_{r_{n+1}})$, we can easily obtain the $\overline{r}$-type of $(\omega, <, W_1, \ldots, W_{r_{n+1}})$, which is one of the elements of the collection C we have to compute.

Among the possible tuples $\overline{W} \in \mathcal{P}(\omega)^{r_{n+1}}$, we have to focus on the ones that define a partition of the set of the indexes ω and that guarantee the following condition: if $Q_h \cap W_{h'}$ is nonempty, for $h \in \{1, ..., s\}$ and $h' \in \{1, ..., r_{n+1}\}$, then the $\overline{k}$-type $\tau_{h'}$ (which is an element of the set $\mathcal{T}^{\overline{k}}(m + k_{n+1})))$ belongs to the $\overline{k}.k_{n+1}$-type σ_h (which is an element of the set $\mathcal{T}^{\overline{k}.k_{n+1}}(m) = \mathcal{P}(\mathcal{T}^{\overline{k}}(m + k_{n+1})))$. With this request we respect the fact that the $\overline{k}$-types $\tau_{h'}$ are generated by the $\overline{k}.k_{n+1}$-type σ_h through some expansion $\overline{R}$. It can be easily seen that these conditions are necessary and sufficient to characterize, among all possible tuples, just the ones we are looking for.

To summarize, we select just those $\overline{r}$-types τ in $\mathcal{T}^{\overline{r}.r_{n+1}}(\omega, <, Q_1, \ldots, Q_s)$ with the following properties: if $Z_1, \ldots, Z_{r_{n+1}}$ denote variables for $W_1, \ldots, W_{r_{n+1}}$, then

1. the formula $Z_1 \cup \ldots \cup Z_{r_{n+1}} = ALL$ is induced by τ, while the formula $NE(Z_{j_1} \cap Z_{j_2})$, for distinct $j_1, j_2 \in \{1, ..., r_{n+1}\}$, is not;

2. if $NE(X_h \cap Z_{h'})$ is induced by τ, then $\tau_{h'} \in \sigma_h$.

Once we have the collection of the $\overline{r}$-types in $\mathcal{T}^{\overline{r}.r_{n+1}}(\omega, <, Q_1, \ldots, Q_s)$ satisfying the above properties, we can easily obtain the set

$$\{\mathcal{T}^{\overline{r}}(\omega, <, Q_1^{\overline{R}}, \ldots, Q_{s'}^{\overline{R}}) : \overline{R} \in \mathcal{P}(\omega)^{k_{n+1}}\},$$

and from this set, by induction, the set $\mathcal{T}^{\overline{k}.k_{n+1}}(\phi_0^\omega(a), \overline{P})$. □

In a similar way, we can prove that the $\overline{k}$ type of the finite square $\phi_0^{n+1}(a)$ only depends on the $\overline{k}$ type of the square $\phi_0^n(a)$ (it suffices to remember how to obtain $\phi_0^{n+1}(a)$ from $\phi_0^n(a)$ and then to adapt the proof of Theorem 13). We have the following theorem.

Theorem 14. *If n, m are such that $\phi_0^n(a) \equiv_{\overline{k}} \phi_0^m(a)$, then $\phi_0^{n+1}(a) \equiv_{\overline{k}} \phi_0^{m+1}(a)$.*

Putting together Theorems 13 and 14, we get the following corollary.

Corollary 15. *The fixed point $\phi_0^\omega(a)$ defining the* FLIP*-predicate is linear.*

Then, decidability follows from Theorem 12.

Theorem 16. *The MSO theory of the infinite square $\phi_0^\omega(a)$ is decidable.*

Finally, since the structure $(\omega, <, \mathrm{FLIP})$ is a projection of $\phi_0^\omega(a)$, decidability of the MSO theory of $(\omega, <, \mathit{FLIP})$ is an easy corollary of Theorem 16.

Corollary 17. *The MSO theory of $(\omega, <, \mathrm{FLIP})$ is decidable.*

6 Conclusions and further work

The aim of this paper was to show that Shelah's composition method can be successfully exploited to decide MSO theories of meaningful expansions of $(\omega, <)$. We first applied it to the class of ultimately type-periodic (unary) predicates, which includes that of unary morphic predicates, and then we generalize it to a suitable class of morphic predicates of higher arity, which includes the binary relation *FLIP*. As for future work, we are trying to establish the generality of the proof method we developed to deal with morphic predicates of higher arity.

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