Directional complexity of the hypercubic billiard

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Abstract

We consider the billiard map in a unit cube of $\mathbb{R}^{d+1}$, and we compute the complexity of a word which codes an orbit in direction ω. Under some hypothesis on the direction, we obtain an exact formula and shows that the order of magnitude is n^d.

1 Introduction

A billiard ball, i.e. a point mass, moves inside a polyhedron P with unit speed along a straight line until it reaches the boundary ∂P, then instantaneously changes direction according to the mirror law, and continues along the new line. Label the sides of P by symbols from a finite alphabet $\mathcal{A}$ whose cardinality equals the number of faces of P. The orbit of a point corresponds to a word in the alphabet $\mathcal{A}$. We define the complexity of this word by the number of words of length n which appears in this infinite word.This complexity depends of the infinite trajectory, we call it the directional complexity, indeed it does not depend on the initial point under suitable hypothesis. We denote it $p(n, \omega)$. How complex is the game of billiard in the hypercube?

For the square (coded with two letters) we obtain Sturmian words and complexity $n + 1$. It is the famous paper of Morse and Hedlund [13], and it has been generalized to any rational polygon by Hubert [10]. It is always linear in n. For an irrational polygon the only general result is that the billiard in a polygon has zero entropy [9] [11], and thus the complexity grows sub-exponentially. For the convex polyhedron the same fact is true [4]. The directional complexity, in the case of the cube (coded with three letters) has been computed by Arnoux, Mauduit, Shiokawa and Tamura [1] (dimension 3). Unfortunately this result was false, and we need some additional hypothesis on the direction, see [3] and [5] for a classification of complexity along the direction. Moreover in [3] the computation has been done in the case of some right prism with tiling polygonal base. The directional complexity is always quadratic in n for those polyhedra, if the direction fulfils some good hyothesis.

In dimension $d + 1 \geq 3$, the computation has been done by Baryshnikov [2] for the hypercube. The complexity is a polynomial on n of degree d.

Here we compute the directional complexity for the hypercube. Our hypothesis are weaker than these of Baryshnikov. The definitions are given in the following section. We obtain:

Theorem 1. *Consider an unit cube of $\mathbb{R}^{d+1}$, we code it by an alphabet with $d+1$ letters. Let ω be a B direction, consider a billiard word in the direction ω, denote the complexity of this word by $p(n, d, \omega)$. Fix $n, d \in \mathbb{N}$, the map $\omega \mapsto p(n, d, \omega)$ is constant on the set of B directions. Moreover if we denote it by $p(n, d)$ we have*

$$p(n + 2, d) - 2p(n + 1, d) + p(n, d) = d(d - 1)p(n, d - 2) \quad \forall\ n, d \in \mathbb{N}.$$

Corollary 2. • *For a B-direction, we have*

$$p(n,d,\omega) = \sum_{i=0}^{min(n,d)} \frac{n!d!}{(n-i)!(d-i)!i!} \quad \forall\ n,d \in \mathbb{N}.$$

We obtain the same formula that Baryshnikov, but our hypothesis on the direction is weaker.
Convention: We assume that $p(n,0) = p(0,d) = 1$ for all integers n,d.
In the two followings sections we recall the usual definitions and previous results.

2 Background

2.1 Combinatorics

For this section a general reference is [12] or [8].

Definition 3. *Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we mean always a factorial extendable language: a language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1a_2\dots a_n$ where $a_i \in \mathcal{A}$ and such that for each $v \in L_n$ there exists $a,b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and for all $v \in L_{n+1}$ if $v = au = u'b$ with $a,b \in \mathcal{A}$ then $u,u' \in L_n$.*
The complexity function $p : \mathbb{N} \to \mathbb{N}$ is defined by $p(n) = card(L_n)$.

First of all we recall a well known result of Cassaigne concerning combinatorics of words [6].

Definition 4. *Let $\mathcal{L}(n)$ be an extendable, factorial language. For any $n \geq 1$ let $s(n) := p(n+1) - p(n)$. For $v \in \mathcal{L}(n)$ let*

$$m_l(v) = card\{u \in \mathcal{A}, uv \in \mathcal{L}(n+1)\},$$
$$m_r(v) = card\{w \in \mathcal{A}, vw \in \mathcal{L}(n+1)\},$$
$$m_b(v) = card\{u \in \mathcal{A}, w \in \mathcal{A}, uvw \in, \mathcal{L}(n+2)\}.$$

A word is call right special if $m_r(v) \geq 2$, left special if $m_l(v) \geq 2$ and bispecial if it is right and left special. Let $\mathcal{BL}(n)$ be the set of the bispecial words.

Cassaigne [6] has shown:

Lemma 5. *Let a language $\mathcal{L}$, then the complexity satisfies*

$$\forall n \geq 1 \quad s(n+1) - s(n) = \sum_{v \in \mathcal{BL}(n)} i(v),$$

where $i(v) = m_b(v) - m_r(v) - m_l(v) + 1$.

For the proof of the lemma we refer to [6] or [7].

2.2 Billiard

In this section we recall some usual definitions, see [14]: Let P be a polyhedron, the billiard map is called T, it is defined on a subset of $\partial P \times \mathbb{PR}^{d+1}$, this space is called the phase space. In the following we identify an element of $\mathbb{PR}^{d+1}$ with a unit vector of $\mathbb{R}^{d+1}$. Moreover if the projection of (m,ω) on ∂P is on a face F we denote $(m,\omega) \in F$.

• We will call face of the cube a face of dimension d. If we use a face of smaller dimension we will precise the dimension.

• Coding. We consider an alphabet associate to the billiard map inside the cube: we associate a letter to each face of the cube such that two parallel faces have the same letter. Let $(F_i)_{i\in\mathbb{Z}}$ be the sequence of faces, then a billiard orbit $(T^n(m,\omega))_{n\in\mathbb{Z}}$ is coded by the word $v=(v_n)_{n\in\mathbb{Z}}$ if and only if

$$T^n(m,\omega)\in F_i \Longleftrightarrow v_n=i.$$

Definition 6. *A direction ω is a minimal direction if for all point m the orbit $(T^n(m,\omega)\cap\partial P)_{n\in\mathbb{Z}}$ is dense in ∂P.*

The following result is classical, see [14] for a proof:

Lemma 7. *Consider the billiard map inside a polyhedron P. If ω is a minimal direction, then the complexity of the word v is independent of the initial point m.*

It allows us to have the following definition:

Definition 8. *Let ω a minimal direction, and m a point of the boundary. Let v be the coding of the orbit of (m,ω) under T. The directional complexity $p(n,\omega,d)$ is the complexity of the word v.*

Moreover we denote $\mathcal{L}(n,d,\omega)$ the set of all words of length n which appear in direction ω.

3 Definitions

Definition 9. *The real numbers $(a_i)_{i\leq n}$ are independent over $\mathbb{Q}$ if and only if*

$$\sum_{i\leq n} r_i a_i=0, r_i\in\mathbb{Q}\Longrightarrow r_i=0\ \forall i\leq n.$$

Definition 10. *Let d be an element of $\mathbb{N}^*$. A vector $\omega=(\omega_i)\in\mathbb{R}^{d+1}$ is called an irrational direction if and only if:*

The real numbers $(\omega_i)_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$.

Definition 11. *Let d be an element of $\mathbb{N}^*$. A vector $\omega=(\omega_i)\in\mathbb{R}^{d+1}$ is called a totally irrational direction if and only if:*

The real numbers $(\omega_i)_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$.
The real numbers $(\omega_i^{-1})_{i\leq d+1}$ are independent over $\mathbb{Q}$.

Definition 12. *Let d be a non negative integer. A vector $\omega=(\omega_i)\in\mathbb{R}^{d+1}$ is called a B direction if and only if:*

The real numbers $(\omega_i)_{1\leq d+1}$ are independent over $\mathbb{Q}$.
For each subset $I\subset\{1\dots d\}$ of cardinality three, the real numbers $(\omega_i^{-1})_{i\in I}$ are independent over $\mathbb{Q}$.

Now we recall the theorem of Baryshnikov [2].

Theorem 13 (Baryshnikov). *Consider an unit cube of $\mathbb{R}^{d+1}$, we code it by an alphabet with $d+1$ letters. Let ω be a totally irrational direction, consider a billiard word in the direction ω, denote the complexity of this word by $p(n,d,\omega)$. Then we have*

$$p(n,d,\omega)=\sum_{i=0}^{min(n,d)}\frac{n!d!}{(n-i)!(d-i)!i!}\quad\forall\ n,d\in\mathbb{N}.$$

Remark 14. *We have the implications:*

$$\omega \text{ totally irrational direction}\Longrightarrow\omega\ B \text{ direction}\Longrightarrow\omega \text{ irrational direction.}$$

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